A DEFECT RELATION FOR HOLOMORPHIC CURVES INTERSECTING HYPERSURFACES

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Abstract. In 1933, H. Cartan proved a defect relation \( \sum_{j=1}^{q} \delta_f(H_j) \leq n + 1 \) for a linearly non-degenerate holomorphic curve \( f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C}) \) and hyperplanes \( H_j, 1 \leq j \leq q \), in \( \mathbb{P}^n(\mathbb{C}) \) in general position. This paper extends it to holomorphic curves intersecting hypersurfaces. In 1979, B. Shiffman conjectured that if \( f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C}) \) is an algebraically non-degenerate holomorphic map, and \( D_1, \ldots, D_q \) are hypersurfaces in \( \mathbb{P}^n(\mathbb{C}) \) in general position, then \( \sum_{j=1}^{q} \delta_f(D_j) \leq n + 1 \). This paper proves this conjecture.

1. Introduction and statements. Let \( f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C}) \) be a linearly non-degenerate holomorphic map, and \( H_j, 1 \leq j \leq q \), be hyperplanes in \( \mathbb{P}^n(\mathbb{C}) \) in general position. In 1933, H. Cartan [Ca] proved the defect relation \( \sum_{j=1}^{q} \delta_f(H_j) \leq n + 1 \). Since then, it has been a long-standing problem to extend Cartan’s result to non-linear hypersurfaces. This paper deals with this problem. To state our result, we first introduce some standard notations in Nevanlinna theory: Let \( f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C}) \) be a holomorphic map. Let \( \tilde{f} = (f_0, \ldots, f_n) \) be a reduced representative of \( f \), where \( f_0, \ldots, f_n \) are entire functions on \( \mathbb{C} \) and have no common zeros. The Nevanlinna-Cartan characteristic function \( T_f(r) \) is defined by

\[
T_f(r) = \frac{1}{2\pi} \int_0^{2\pi} \log \|\tilde{f}(re^{i\theta})\| d\theta
\]

where

\[
\|\tilde{f}(z)\| = \max\{|f_0(z)|, \ldots, |f_n(z)|\}.
\]

The above definition is independent, up to an additive constant, of the choice of the reduced representation of \( f \). Let \( D \) be a hypersurface in \( \mathbb{P}^n(\mathbb{C}) \) of degree \( d \). Let \( Q \) be the homogeneous polynomial (form) of degree \( d \) defining \( D \). The proximity function \( m_f(r, D) \) is defined as

\[
m_f(r, D) = \int_0^{2\pi} \log \frac{\|\tilde{f}(re^{i\theta})\|^d}{Q(\tilde{f}(re^{i\theta}))} d\theta.
\]

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The above definition is independent, up to an additive constant, of the choice of the reduced representation of $f$. To define the counting function, let $n_f(r, D)$ be the number of zeros of $Q \circ \tilde{f}$ in the disk $|z| < r$, counting multiplicity. The counting function is then defined by

$$N_f(r, D) = \int_0^r \frac{n_f(t, D) - n_f(0, D)}{t} dt + n_f(0, D) \log r.$$ 

The Poisson-Jensen formula implies:

**First Main Theorem.** Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map, and let $D$ be a hypersurface in $\mathbb{P}^n(\mathbb{C})$ of degree $d$. If $f(\mathbb{C}) \not\subset D$, then for every real number $r$ with $0 < r < \infty$

$$m_f(r, D) + N_f(r, D) = dT_f(r) + O(1),$$

where $O(1)$ is a constant independent of $r$.

Hypersurfaces $D_1, \ldots, D_q$, $q > n$, in $\mathbb{P}^n(\mathbb{C})$ are said to be in general position if $\bigcap_{k=1}^{n+1} \text{Supp}(D_{j_k}) = \emptyset$ for any distinct $j_1, \ldots, j_{n+1}$. In this paper, the following Shiffman’s Conjecture is proved.

**Main Theorem.** Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be an algebraically non-degenerate holomorphic map, and let $D_j, 1 \leq j \leq q$, be hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ of degree $d_j$ in general position. Then for every $\epsilon > 0$,

$$\sum_{j=1}^q d_j^{-1} m_f(r, D_j) \leq (n + 1 + \epsilon)T_f(r),$$

where the inequality holds for all $r \in (0, +\infty)$ except for a possible set $E$ with finite Lebesgue measure.

Define the defect

$$\delta_f(D) = \liminf_{r \to +\infty} \frac{m_f(r, D)}{dT_f(r)}.$$

Then we have the following defect relation:

**Corollary (Defect Relation).** Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be an algebraically non-degenerate holomorphic map, and let $D_1, \ldots, D_q$ be hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ in general position. Then we have

$$\sum_{j=1}^q \delta_f(D_j) \leq n + 1.$$
Note that by the standard process of averaging over the complex lines in the complex space \( \mathbb{C}^m \), one can easily extend our results to holomorphic map \( f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C}) \).

The strongest previously known defect relation for nonlinear hypersurfaces was a theorem of Eremenko and Sodin ([ES], 1992) that under the same assumption on the hypersurfaces,

\[
\sum_{j=1}^{q} \delta_f(D_j) \leq 2n
\]

for all nonconstant curves \( f \) whose image is not contained in \( \bigcup_{j=1}^{q} D_j \). Earlier, B. Shiffman [Shi] and A. Biancofiore [B] proved

\[
\sum_{j=1}^{q} \delta_f(D_j) \leq 2n
\]

for a special class of nonconstant holomorphic mappings \( f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C}) \). In [Siu], Y.-T. Siu obtained

\[
\sum_{j=1}^{q} \delta_f(D_j) \leq en
\]

using new techniques developed by Siu-Yeung and M. McQuillan. It is not clear whether the result of Eremenko and Sodin can be derived from our Main Theorem.

The sharp defect relation for nonlinear hypersurfaces was conjectured by Ph. Griffiths as follows:

**Conjecture (Griffiths’ conjecture).** Let \( f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C}) \) be an algebraically non-degenerate holomorphic map, and let \( D_1, \ldots, D_q \) be hypersurfaces of degree \( d \) in \( \mathbb{P}^n(\mathbb{C}) \) in general position. Then we have

\[
\sum_{j=1}^{q} \delta_f(D_j) \leq (n + 1)/d.
\]

The proof of our Main Theorem here is motivated by the analogy between Nevanlinna theory and Diophantine approximation, discovered by C. Osgood, P. Vojta and S. Lang, etc. It is particularly inspired by a recent manuscript (see [CZ]) by P. Corvaja and U. Zannier, where they proved the counterpart of this result in Diophantine approximation, namely Schmidt’s subspace theorem for homogeneous polynomial forms.

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2. Some lemmas. In this section, we state several lemmas from [CZ], regarding some facts from commutative algebra. For the sake of completeness, we also include the proofs.

**Lemma 2.1.** Let $\phi_1, \ldots, \phi_m$ be homogeneous polynomials in $\mathbb{C}[X_0, \ldots, X_n]$. Assume that they define a subvariety of $\mathbb{P}^n$ of dimension $n - m$. Then \{ $\phi_1, \ldots, \phi_m$ \} is a regular sequence, that is, for $i = 1, \ldots, m$ the element $\phi_i$ is not a divisor of the zero in the ring $\mathbb{C}[X_0, \ldots, X_n]_{(\phi_1, \ldots, \phi_{i-1})}$.

**Proof.** This is a well-known fact from the theory of Cohen-Macaulay rings. $\square$

Throughout of this paper, we shall use the lexicographic ordering on the $m$-tuples $(i_1, \ldots, i_m) \in \mathbb{N}^m$ of natural numbers. Namely, $(j_1, \ldots, j_m) > (i_1, \ldots, i_m)$ iff for some $b \in \{1, \ldots, m\}$ we have $j_l = i_l$ for $l < b$ and $j_b > i_b$.

**Lemma 2.2.** Let $R$ be a commutative ring and let $\phi_1, \ldots, \phi_m$ be a regular sequence in $R$, generating the ideal $I \subset R$. Suppose that for some $q, q_1, \ldots, q_h \in R$ we have an equation

$$
\phi_1^{i_1} \cdots \phi_m^{i_m} q = \sum_{r=1}^h \phi_1^{j_1(r)} \cdots \phi_m^{j_m(r)} q_r
$$

where $(j_1(r), \ldots, j_m(r)) > (i_1, \ldots, i_m)$ (in the lexicographic order) for $r = 1, \ldots, h$. Then $q \in I$.

**Proof.** We argue by induction on $m$. Since $\phi_1$ is not a zero divisor in $R$, the assertion is trivial for $m = 1$. Assume $m > 1$ and the lemma is true up to $m - 1$. Renumbering the indices $1, \ldots, h$ we may assume that $j_1(r) > i_1$ for $r = 1, \ldots, s$ (possibly $s = 0$ or $s = h$) and that $j_1(r) = i_1$ for $r = s + 1, \ldots, h$ (the case $j_1(r) < i_1$ is excluded since $(j_1(r), \ldots, j_m(r)) > (i_1, \ldots, i_m)$). Since $\phi_1$ is not a zero divisor in $R$ we may then write

$$
\phi_1^{i_2} \cdots \phi_m^{i_m} q = \phi_1^{i_1} \sigma + \sum_{r=s+1}^h \phi_2^{j_2(r)} \cdots \phi_m^{j_m(r)} q_r,
$$
where $\sigma \in \mathbb{R}$. We now reduce modulo $\phi_1$, denoting the reduction with a bar, and work in the ring $R' = R/(\phi_1)$. We obtain

$$\bar{\phi}_2^{j_2} \cdots \bar{\phi}_m^{j_m} \bar{q} = \sum_{r=s+1}^{h} \bar{\phi}_2^{j_2(r)} \cdots \bar{\phi}_m^{j_m(r)} \bar{q}_r.$$ 

Note that $(j_2(r), \ldots, j_m(r)) > (i_2, \ldots, i_m)$ for $r = s+1, \ldots, m$ and that $\{\bar{\phi}_2, \ldots, \bar{\phi}_m\}$ is a regular sequence in $R'$. We may thus apply the inductive assumption with $m-1$ in place of $m$ and $R'$ in place of $R$. We obtain that $\bar{q}$ lies in the ideal of $R'$ generated by $\bar{\phi}_2, \ldots, \bar{\phi}_m$, i.e. that $q \in I$, as required.

Let $V_N$ be the space of homogeneous polynomials of degree $N$ in $\mathbb{C}[X_0, \ldots, X_n]$. We have the following lemma (see [CZ], Lemma 2.3):

**Lemma 2.3.** Let $\phi_1, \ldots, \phi_n$ be homogeneous polynomials in $\mathbb{C}[X_0, \ldots, X_n]$ and assume that they define a subvariety of $\mathbb{P}^n(\mathbb{C})$ of dimension $0$. Then, for all large $N$,

$$\dim \frac{V_N}{(\phi_1, \ldots, \phi_n) \cap V_N} = \deg \phi_1 \cdots \deg \phi_n.$$ 

**Proof.** Essentially we have here the Bezout Theorem for $\mathbb{P}^n$. For a proof, recall first that it is a classical fact from the theory of Hilbert polynomials that, under our assumption on the dimension, $\dim \frac{V_N}{(\phi_1, \ldots, \phi_n)}$ is constant for large $N$, equal to the degree of the variety defined by $\phi_1, \ldots, \phi_n$ (see [H, Ch. I.7]) which is just the product of the degrees of $\phi_i$ (see e.g. [Sh, Ch IV]).

**3. Proof of the Main Theorem.** To prove the Main Theorem, we need the following general form of the Second Main Theorem for holomorphic curves intersecting hyperplanes. The theorem is stated and proved in [Ru] (see Theorem 2.1 in [Ru]).

**Theorem 3.1.** Let $f = [f_0 : \cdots : f_n] : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map whose image is not contained in any proper linear subspace. Let $H_1, \ldots, H_q$ be arbitrary hyperplanes in $\mathbb{P}^n(\mathbb{C})$. Let $L_j, 1 \leq j \leq q$, be the linear forms defining $H_1, \ldots, H_q$. Then, for every $\epsilon > 0$,

$$\int_0^{2\pi} \max_K \log \prod_{j \in K} \frac{\|f(re^{i\theta})\|}{\|L_j(f(re^{i\theta}))\|} \frac{d\theta}{2\pi} \leq (n + 1 + \epsilon)T_f(r),$$

where the inequality holds for all $r$ outside of a set $E$ with finite Lebesgue measure, the maximum is taken over all subsets $K$ of $\{1, \ldots, q\}$ such that the linear forms
$L_j, j \in K$, are linearly independent, and $\|L_j\|$ is the maximum of the absolute values of the coefficients in $L_j$.

We now prove our Main Theorem.

**Proof of the Main Theorem.** Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map. Let $\tilde{f} = (f_0, \ldots, f_n)$ be a reduced representative of $f$, where $f_0, \ldots, f_n$ are entire functions on $\mathbb{C}$ and have no common zeros. For simplicity, we just write $\tilde{f}$ as $f$. Let $D_1, \ldots, D_q$ be hypersurfaces in $\mathbb{P}^n(\mathbb{C})$, located in general position. Let $Q_j, 1 \leq j \leq q$, be the homogeneous polynomials in $\mathbb{C}[X_0, \ldots, X_n]$ of degree $d_j$ defining $D_j$. Replacing $Q_j$ by $Q_j^{d/d_j}$ if necessary, where $d$ is the l.c.m of $d_j$'s, we can assume that $Q_1, \ldots, Q_q$ have the same degree of $d$.

Given $z \in \mathbb{C}$, there exists a renumbering $\{i_1, \ldots, i_q\}$ of the indices $\{1, \ldots, q\}$ such that

$$|Q_{i_1} \circ f(z)| \leq |Q_{i_2} \circ f(z)| \leq \cdots \leq |Q_{i_q} \circ f(z)|.$$  \hspace{1cm} (3.1)

Since $Q_1, \ldots, Q_q$ are in general position, by Hilbert's Nullstellensatz (cf. [W]) that for any integer $k, 0 \leq k \leq n$, there is an integer $m_k \geq d$ such that

$$x_k^{m_k} = \sum_{j=1}^{n+1} b_{jk}(x_0, \ldots, x_n)Q_j(x_0, \ldots, x_n),$$

where $b_{jk}, 1 \leq j \leq n+1, 0 \leq k \leq n$, are the homogeneous forms with coefficients in $\mathbb{C}$ of degree $m_k - d$. So

$$|f_k(z)|^{m_k} \leq c_1 \|f(z)\|^{m_k - d} \max\{|Q_{i_1} (f)(z)|, \ldots, |Q_{i_{n+1}} (f)(z)|\},$$

where $c_1$ is a positive constant depends only on the coefficients of $b_{jk}, 1 \leq i \leq n+1, 0 \leq k \leq n$, thus depends only on the coefficients of $Q_i, 1 \leq i \leq n+1$. Therefore,

$$\|f(z)\|^d \leq c_1 \max\{|Q_{i_1} (f)(z)|, \ldots, |Q_{i_{n+1}} (f)(z)|\}. $$

By (3.1) and (3.2),

$$\prod_{j=1}^{q} \frac{\|f(z)\|^d}{|Q_{i_j}(f)(z)|} \leq c_1^{q-n} \prod_{k=1}^{n} \frac{\|f(z)\|^d}{|Q_k(f)(z)|}. $$

Hence, by the definition,

$$\sum_{j=1}^{q} m_j(r, Q_j) \leq \int_0^{2\pi} \max_{\{i_1, \ldots, i_n\}} \left\{ \log \prod_{k=1}^{n} \frac{\|f(re^{i\theta})\|^d}{|Q_k(f)(re^{i\theta})|} \right\} \frac{d\theta}{2\pi} + (q - n) \log c_1.$$  \hspace{1cm} (3.3)
Now pick \( n \) distinct polynomials \( \gamma_1, \ldots, \gamma_n \in \{Q_1, \ldots, Q_q\} \). By the “in general position” assumption, they define a subvariety of \( \mathbb{P}^n \) of dimension 0. For a fixed big integer \( N \), which will be chosen later, denote by \( V_N \) the space of homogeneous polynomials in \( \mathbb{C}[X_0, \ldots, X_n] \) of degree \( N \). Arrange, by the lexicographic order, the \( n \)-tuples \( (i) = (i_1, \ldots, i_n) \) of non-negative integers such that \( \sigma(i) := \sum_j i_j \leq N/d \). Define the spaces \( W_{(i)} = W_{N,(i)} \) by

\[
W_{(i)} = \sum_{(e) \geq (i)} \gamma_1^{e_1} \cdots \gamma_n^{e_n} V_{N-d\sigma(e)}.
\]

Plainly \( W_{(0, \ldots, 0)} = V_N \) and \( W_{(i)} \supset W_{(i')} \) if \( (i') \geq (i) \), so the \( W_{(i)} \) in fact define a filtration of \( V_N \).

Our next step is to investigate quotients between consecutive spaces in the filtration. Suppose that \( (i') \) follows \( (i) \) in the ordering. We have the following lemma:

**Lemma 3.2.** There is an isomorphism

\[
\frac{W_{(i)}}{W_{(i')}} \cong \frac{V_{N-d\sigma(i)}}{(\gamma_1, \ldots, \gamma_n) \cap V_{N-d\sigma(i)}}.
\]

**Proof:** Define a vector space homomorphism \( \phi : V_{N-d\sigma(i)} \to W_{N,(i)} \) as follows: \( q(X_0, \ldots, X_n) \) is a polynomial in \( V_{N-d\sigma(i)} \), we define \( \phi(q) \) as the class of \( \gamma_1^{e_1} \cdots \gamma_n^{e_n} q \) (which belongs to \( W_{(i)} \)) modulo \( W_{(i')} \). By our definition of the spaces \( W_{(e)} \), this homomorphism is surjective. To find its kernel, suppose that \( q \in \ker \phi \). This means that \( \gamma_1^{e_1} \cdots \gamma_n^{e_n} q \) lies in \( \sum_{(e) > (i)} \gamma_1^{e_1} \cdots \gamma_n^{e_n} V_{N-d\sigma(e)} \). Then we may write

\[
\gamma_1^{i_1} \cdots \gamma_n^{i_n} q = \sum_{(e) > (i)} \gamma_1^{e_1} \cdots \gamma_n^{e_n} q_{(e)},
\]

for elements \( q_{(e)} \in V_{N-d\sigma(e)} \). By Lemma 2.2, \( q \) lies in the ideal generated by \( \gamma_1, \ldots, \gamma_n \). Therefore \( q = \sum_{j=1}^n \alpha_j \gamma_j \) where \( \alpha_j, 1 \leq j \leq n \) are homogeneous with \( \deg \alpha_j = \deg q - d \). Then \( \alpha_j \in V_{N-d\sigma(i)+1} \). Thus we see that \( \gamma_1^{i_1} \cdots \gamma_n^{i_n} q \) is a sum of terms in \( W_{(i')} \), which concludes the proof of Lemma.

We now use the above lemma to evaluate the dimensions, denoted by \( \Delta_{(i)} \), of the quotients of successive spaces in the filtration. Applying Lemma 3.2 and Lemma 2.3 implies:

**Lemma 3.3.** There exists an integer \( N_0 \) dependent only on \( \gamma_1, \ldots, \gamma_n \) such that, in the above notation,

\[
\Delta_{(i)} := \dim \frac{W_{(i)}}{W_{(i')}} = d^n
\]
provided $d\sigma(i) < N - N_0$. Also, for the remaining $n$-tuples $(i)$, $\dim \frac{W_{\sigma(i)}}{W_{\sigma(i)'}}$ is bounded (by $\dim V_{N_0}$).

Set $M = M_N := \dim V_N$. We now chose a suitable basis $\{\psi_1, \ldots, \psi_M\}$ for $V_N$ is the following way. We start with the last nonzero $W_{(i)}$ and pick any basis of it. Then we continue inductively as follows: suppose $(i') > (i)$ are consecutive $n$-tuples such that $d\sigma(i), d\sigma(i') \leq N$ and assume that we have chosen a basis of $W_{(i')}$. It follows directly from the definition that we may pick representatives in $W_{(i')} / W_{(i)}$, of the form $\gamma_1^i \cdots \gamma_n^i q$, where $q \in V_{N - d\sigma(i)}$. We extend the previously constructed basis in $W_{(i')} / W_{(i)}$ by adding these representatives.

We extend the previously constructed basis in $W_{(i')}$ by adding these representatives. In particular, we have obtained a basis for $W_{(i)}$ and our inductive procedure may go on unless $W_{(i)} = V_N$, in which case we stop. In this way, we obtain a basis $\{\psi_1, \ldots, \psi_M\}$ for $V_N$.

We now estimate $\log \prod_{t=1}^M |\psi_t (f)(z)|$. Let $\psi$ be an element of the basis, constructed with respect to $W_{(i')}/W_{(i)}$, so we may write $\psi = \gamma_1^i \cdots \gamma_n^i q$, where $q \in V_{N - d\sigma(i)}$. Then we have a bound

$$\begin{align*}
|\psi(f)(z)| &\leq |\gamma_1(f)(z)|^{|i_1|} \cdots |\gamma_n(f)(z)|^{|i_n|} |g(f)(z)| \\
&\leq c_2 |\gamma_1(f)(z)|^{|i_1|} \cdots |\gamma_n(f)(z)|^{|i_n|} \|f(z)\|^{N - d\sigma(i)},
\end{align*}$$

where $c_2$ is a positive constant depends only on $\psi$, not on $f$ and $z$. Observe that there are precisely $\Delta_{(i)}$ such functions $\psi$ in our basis. Hence, taking the product over all functions in the basis, we get, after taking logarithms,

$$\begin{align*}
\log \prod_{t=1}^M |\psi_t (f)(z)| &\leq \sum_{(i)} \Delta_{(i)} \left( i_1 \log |\gamma_1(f)(z)| + \cdots + i_n \log |\gamma_n(f)(z)| \right) \\
&\quad + \log \|f(z)\| \left( \sum_{(i)} \Delta_{(i)} (N - d\sigma(i)) \right) + c_3,
\end{align*}$$

where $c_3$ depends only on the $\psi$’s, not on $f$ and $z$. Here the summation is taken over the $n$-tuples with $\sigma(i) \leq N/d$.

We now estimate the sums. First,

$$M = \frac{(N + n)!}{N! n!} = \frac{N^n}{n!} + O(N^{n-1}).$$

Second, since the number of nonnegative integer $m$-tuples with sum $\leq T$ is equal to the number of non-negative integer $(m + 1)$-tuples with sum exactly $T \in \mathbb{Z}$, which is $\binom{T + m}{m}$, and since the sum below is independent of $j$, we have that, for
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N divisible by d and for every j,

\[ \sum_{(i)} i_j = \frac{1}{n+1} \sum_{(i)} \sum_{j=1}^{n+1} i_j = \frac{1}{n+1} \sum_{(i)} \frac{N}{d} \]

\[ = \frac{1}{n+1} \left( \frac{N/d + n}{n} \right) \frac{N}{d} = \frac{N^{n+1}}{d^{n+1}(n+1)!} + O(N^n), \]

where the sum \( \sum_{(i)} \) is taken over the nonnegative integer \((n+1)-\)tuples with sum exactly \(N/d\). Combing (3.6) and Lemma 3.3, we have, for every \(1 \leq j \leq n\),

\[ \sum_{(i)} i_j \Delta(i) = \frac{N^{n+1}}{d(n+1)!} + O(N^n), \]

where again the summations are taken over the \(n\)-tuples with sum \(\leq N/d\); also the various constants in the “\(O\)” terms depend only on the original data, \(\gamma_1, \ldots, \gamma_n\), hence only on \(Q_1, \ldots, Q_q\), not on \(f\) and \(z\). (3.7) implies that

\[ \sum_{(i)} \Delta(i) \sigma(i) = \frac{nN^{n+1}}{d(n+1)!} + O(N^n). \]

Therefore, combining (3.4), (3.5), (3.7) and (3.8) yields

\[ \log \prod_{j=1}^{M} |\psi_j(f)(z)| \leq \left( \log \prod_{j=1}^{n} |\gamma_j(f)(z)| \right) \frac{N^{n+1}}{d(n+1)!} (1 + O(N^{-1})) \]

\[ + (\log \|f(z)\|) \frac{nN^{n+1}}{d(n+1)!} (1 + O(N^{-1})) + c_3, \]

where \(c_3\) and the various constants in the “\(O\)” terms depend only on \(Q_1, \ldots, Q_q\), not on \(f\) and \(z\).

Now let \(\phi_1, \ldots, \phi_M\) be a fixed basis of \(V_N\). Then \(\{\psi_1, \ldots, \psi_M\}\) can be written as linear forms \(L_1, \ldots, L_M\) in \(\phi_1, \ldots, \phi_M\) so that \(\psi_i(f) = L_i(F)\), where \(F = [\phi_1(f) : \cdots : \phi_M(f)]\). The linear forms \(L_1, \ldots, L_M\) are linearly independent, and we know, from the assumption of algebraically nondegeneracy of \(f\), that \(F\) is linearly nondegenerate. From (3.9),

\[ \log \prod_{i=1}^{M} |L_i(F)(z)| \leq \left( \log \prod_{j=1}^{n} |\gamma_j(f)(z)| \right) \frac{N^{n+1}}{d(n+1)!} (1 + O(N^{-1})) \]

\[ + (\log \|f(z)\|) \frac{nN^{n+1}}{(n+1)!} (1 + O(N^{-1})) + c_3 \]
\[
= \left[ \log \prod_{j=1}^{n} \frac{\gamma_j(f(z))}{\|f(z)\|^d} + (n+1)d \log \|f(z)\| \right] \frac{N^{n+1}}{d(n+1)!}(1+O(N^{-1})) + c_3.
\]

This implies that

\[
\log \prod_{j=1}^{n} \frac{\|f(z)\|^d}{\gamma_j(f(z))} \leq \frac{d(n+1)!}{N^{n+1}(1+O(N^{-1}))} \left[ \log \prod_{i=1}^{M} \frac{\|F(z)\|}{|L_i(F)(z)|} - M \log \|F(z)\| + c_3 \right] + d(n+1) \log \|f(z)\|.
\]

Since the various constants in the "O" term above depend only on \(Q_1, \ldots, Q_q\), not on \(f\) and \(z\), for the \(\epsilon > 0\) given in the Main Theorem, take \(N\) large enough such that

\[
\frac{d(n+1)!}{N^n(1+O(N^{-1}))} < \epsilon/2.
\]

Fix such an \(N\). We now continue our proof. With such chosen \(N\), (3.10) becomes

\[
\log \prod_{j=1}^{n} \frac{\|f(z)\|^d}{\gamma_j(f(z))} \leq \frac{\epsilon}{2N} \left[ \log \prod_{i=1}^{M} \frac{\|F(z)\|}{|L_i(F)(z)|} - M \log \|F(z)\| + c_3 \right] + d(n+1) \log \|f(z)\|.
\]

Since there are only finitely many choices \(\{\gamma_1, \ldots, \gamma_n\} \subset \{Q_1, \ldots, Q_q\}\), we have a finite collection of linear forms \(L_1, \ldots, L_u\). (3.11) thus implies that

\[
\int_0^{2\pi} \max_{\{l_1, \ldots, l_n\}} \left\{ \log \prod\limits_{k=1}^{n} \frac{\|f(re^{i\theta})\|^d}{Q_k(f(re^{i\theta}))} \right\} d\theta 2\pi
\]

\[
\leq \frac{\epsilon}{2N} \left[ \int_0^{2\pi} \max_{j \in K} \log \prod_{j \in K} \frac{\|F(re^{i\theta})\|\|L_j\|}{|L_j(F)(re^{i\theta})|} d\theta 2\pi - M \int_0^{2\pi} \log \|F(re^{i\theta})\| d\theta 2\pi + c_4 \right] + d(n+1) \int_0^{2\pi} \log \|f(re^{i\theta})\| d\theta 2\pi
\]

\[
= \frac{\epsilon}{2N} \left[ \int_0^{2\pi} \max_{j \in K} \log \prod_{j \in K} \frac{\|F(re^{i\theta})\|\|L_j\|}{|L_j(F)(re^{i\theta})|} d\theta 2\pi - MT_F(r) + c_4 \right] + d(n+1)T_f(r),
\]
where max$_K$ is taken over all subsets $K$ of $\{1, \ldots, u\}$ such that linear forms $L_j, j \in K$, are linearly independent, and $c_4$ is a constant independent of $r$. Applying Theorem 3.1 with $\epsilon = 1$ to holomorphic map $F$ and linear forms $L_1, \ldots, L_u$, we obtain that

$$\int_0^{2\pi} \max_K \log \prod_{j \in K} \frac{\|F(re^{i\theta})\| \|L_j\|}{|L_j(F(re^{i\theta})|} \frac{d\theta}{2\pi} \leq (M + 1)T_F(r)$$

(3.13)

holds for all $r$ outside of a set $E$ with finite Lebesgue measure. Obviously, $T_F(r) \leq NT_f(r) + c_5$, so combining (3.3), (3.12) and (3.13), we have that

$$\sum_{j=1}^{q} m_f(r, Q_j) \leq d(n + 1)T_f(r) + \frac{\epsilon}{2N}(T_f(r) + c_4) + (q - n) \log c_1$$

$$\leq d(n + 1)T_f(r) + \frac{\epsilon}{2N}(NT_f(r) + c_4 + c_5) + (q - n) \log c_1$$

$$\leq (d(n + 1) + \epsilon/2)T_f(r) + C$$

holds for all $r$ outside of a set $E$ with finite Lebesgue measure, where $C$ is a constant, independent of $r$. Take $r$ large enough so we can make $C \leq (\epsilon/2)T_f(r)$. Thus we have

$$\sum_{j=1}^{q} m_f(r, D_j) \leq (n + 1 + \epsilon)dT_f(r)$$

where the inequality holds for all $r \in (0, +\infty)$ except for a possible set $E$ with finite Lebesgue measure. Here we note that each exceptional set $E$ may be different each time, but it still has finite Lebesgue measure. This finishes the proof of the Main Theorem.

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REFERENCES

[Ca] H. Cartan, Sur les zeros des combinaisons lineairies de $p$ fonctions holomorphes donnees, Mathematatica (Cluj) 7 (1933), 80–103.


