Then

$$\begin{aligned}\int_C \sin(\pi x)\,dy - \cos(\pi y)\,dz &= \sum_{i=1}^{3} \int_{C_i} \sin(\pi x)\,dy - \cos(\pi y)\,dz \\ &= \int_0^1 \sin[\pi(1-t)]\,dt - \cos(\pi t)\cdot 0 \\ &\quad + \int_0^1 \sin(\pi\cdot 0)(-dt) - \cos[\pi(1-t)]\,dt \\ &\quad + \int_0^1 \sin \pi t\cdot 0 - \cos(\pi\cdot 0)(-dt) \\ &= \int_0^1 \sin[\pi(1-t)]\,dt - \int_0^1 \cos[\pi(1-t)]\,dt + \int_0^1 dt \\ &= -\frac{1}{\pi}\{-\cos[\pi(1-t)]\}\Big|_0^1 - \left(-\frac{1}{\pi}\right)\{\sin[\pi(1-t)]\}\Big|_0^1 + 1 \\ &= \frac{1}{\pi}(\cos 0 - \cos\pi) + \frac{1}{\pi}(\sin 0 - \sin\pi) + 1 \\ &= \frac{1}{\pi}[1-(-1)] + \frac{1}{\pi}(0-0) + 1 = \frac{2}{\pi} + 1. \ \blacktriangle\end{aligned}$$

Exercises for Section 18.1

1. Calculate the work which is done by the force field $\mathbf{\Phi}(x, y, z) = x\mathbf{i} + y\mathbf{j}$ when a particle is moved along the path $(3t^2, t, 1)$, $0 \leqslant t \leqslant 1$.
2. Find the work done by the force field in Exercise 1 when a particle is moved along the straight line segment from $(0, 0, 1)$ to $(3, 1, 1)$.
3. Find the work which is done by the force field $\mathbf{\Phi}(x, y) = (x^2 + y^2)(\mathbf{i} + \mathbf{j})$ around the loop $(x, y) = (\cos t, \sin t)$, $0 \leqslant t \leqslant 2\pi$.
4. Find the work done by the force field in Exercise 3 around the loop $(x, y) = (1 + \cos t, 1 + \sin t)$, $0 \leqslant t \leqslant 2\pi$.
5. Suppose that you pick up a unit mass which was at rest at $(1, 0, 0)$ and carry it to $(1, 0, 1)$ along the path $(1, 0, t)$ under the field $xy\mathbf{i} + (x + y)\mathbf{k}$. If you leave the particle with velocity vector $\mathbf{i} + 2\mathbf{j}$ at the end of the trip, how much work have you done?
6. Do as in Exercise 5, except that the particle is left at rest at the end of the trip.
7. Show that if a particle is moved along the closed curve $(\cos t, \sin t, 0)$, $0 \leqslant t \leqslant 2\pi$, then the force field in Example 1 does a nonzero amount of work on the particle. How much is the work?
8. Show that if a particle is moved along a closed curve (that is, $\boldsymbol{\sigma}(t_1) = \boldsymbol{\sigma}(t_2)$), then the work done on it by the gravitational field in Example 3 is zero.

Let $\mathbf{\Phi}(x, y) = [1/(x^2 + y^2)](-y\mathbf{i} + x\mathbf{j})$ be a force field in the plane (minus the origin). Compute the work done by this force along each of the paths in Exercises 9–12.

9. $(\cos t, \sin t)$; $0 \leqslant t \leqslant \pi$
10. $(\cos t, -\sin t)$; $0 \leqslant t \leqslant \pi$
11. $(\cos t, \sin t)$; $0 \leqslant t \leqslant 2\pi$
12. $(-\cos t, \sin t)$; $0 \leqslant t \leqslant 2\pi$

In Exercises 13–20, evaluate the integral of the given vector field $\mathbf{\Phi}$ along the given path.

13. $\boldsymbol{\sigma}(t) = (\sin t, \cos t, t)$, $0 \leqslant t \leqslant 2\pi$, $\mathbf{\Phi}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.
14. $\boldsymbol{\sigma}(t) = (t, t, t)$; $0 \leqslant t \leqslant 1$, $\mathbf{\Phi}(x, y, z) = x\mathbf{i} - y\mathbf{j} + z\mathbf{k}$.
15. $\boldsymbol{\sigma}(t) = (\cos t, \sin t, 0)$; $0 \leqslant t \leqslant \pi/2$, $\mathbf{\Phi}(x, y, z) = x\mathbf{i} - y\mathbf{j} + z\mathbf{k}$.
16. $\boldsymbol{\sigma}(t) = (\cos t, \sin t, 0)$; $0 \leqslant t \leqslant \pi/2$, $\mathbf{\Phi}(x, y, z) = x\mathbf{i} - y\mathbf{j} + 2\mathbf{k}$.
17. $\boldsymbol{\sigma}(t) = (\sin t, t^2, t)$; $0 \leqslant t \leqslant 2\pi$, $\mathbf{\Phi}(x, y, z) = \sin z\mathbf{i} + \cos\sqrt{y}\,\mathbf{j} + x^3\mathbf{k}$.
18. $\boldsymbol{\sigma}(t) = (\cos t, \sec t, \tan t)$; $-\pi/4 \leqslant t \leqslant \pi/4$, $\mathbf{\Phi}(x, y, z) = xz\mathbf{i} + xy\mathbf{j} + yz\mathbf{k}$.
19. $\boldsymbol{\sigma}(t) = ((1 + t^2)^2, 1, t)$; $0 \leqslant t \leqslant 1$, $\mathbf{\Phi}(x, y, z) = [1/(z^2 + 1)]\mathbf{i} + x(1 + y^2)\mathbf{j} + e^y\mathbf{k}$.
20. $\boldsymbol{\sigma}(t) = 3t\mathbf{i} + (t - 1)\mathbf{j} + t^2\mathbf{k}$; $0 \leqslant t \leqslant 1$, $\mathbf{\Phi}(x, y, z) = (x^2 + x)\mathbf{i} + \dfrac{x - y}{x + y}\mathbf{j} + (z - z^3)\mathbf{k}$.

Let $\mathbf{\Phi}(x, y, z) = x^2\mathbf{i} - xy\mathbf{j} + \mathbf{k}$. Evaluate the line integral of $\mathbf{\Phi}$ along each of the curves in Exercises 21–24.

21. The straight line joining $(0, 0, 0)$ to $(1, 1, 1)$.
22. The circle of radius 1, center at the origin and lying in the yz plane, traversed counterclockwise as viewed from the positive x axis.
23. The parabola $z = x^2$, $y = 0$, between $(-1, 0, 1)$ and $(1, 0, 1)$.
24. The straight line between $(-1, 0, 1)$ and $(1, 0, 1)$.
25. Let C be parametrized by $x = \cos^3\theta$, $y = \sin^3\theta$, $z = \theta$, $0 \leqslant \theta \leqslant 7\pi/2$. Evaluate the integral $\int_C \sin z\,dx + \cos z\,dy - (xy)^{1/3}\,dz$.

26. Evaluate $\int_C x^2\,dx + xy\,dy + dz$, where C is parametrized by $\boldsymbol{\sigma}(t) = (t, t^2, 1)$, $0 \leqslant t \leqslant 1$.
27. Evaluate $\int_C \boldsymbol{\Phi}(\mathbf{r}) \cdot d\mathbf{r}$, where $\boldsymbol{\Phi}(x, y, z) = \sin z\mathbf{i} + \cos\sqrt{y}\,\mathbf{j} + x^3\mathbf{k}$ and C is the line segment from $(1,0,0)$ to $(0,0,3)$.
28. Evaluate $\int_C e^{x+y-z}(\mathbf{i} + \mathbf{j} - \mathbf{k}) \cdot d\mathbf{r}$, where C is the path $(\ln t, t, t)$ for $2 \leqslant t \leqslant 4$.

The *line integral of a scalar function f* along a parametric curve $\boldsymbol{\sigma}(t)$, $t_1 \leqslant t \leqslant t_2$, is defined by

$$\int_{t_1}^{t_2} f(\boldsymbol{\sigma}(t))\|\boldsymbol{\sigma}'(t)\|\,dt.$$

Note that if $f = 1$, this is just the arc length of the curve. Evaluate the line integrals of the functions along the indicated curves in Exercises 29–34.

29. $f(x, y, z) = x + y + yz$, where $\boldsymbol{\sigma}(t) = (\sin t, \cos t, t)$; $0 \leqslant t \leqslant 2\pi$.
30. $f(x, y, z) = x + \cos^2 z$, $\boldsymbol{\sigma}$ as in Exercise 29.
31. $f(x, y, z) = x\cos z$, $\boldsymbol{\sigma}(t) = t\mathbf{i} + t^2\mathbf{j}$; $0 \leqslant t \leqslant 1$.
32. $f(x, y, z) = \exp\sqrt{z}$, $\boldsymbol{\sigma}(t) = (1, 2, t^2)$; $0 \leqslant t \leqslant 1$.
33. $f(x, y, z) = yz$, $\boldsymbol{\sigma}(t) = (t, 3t, 2t)$; $1 \leqslant t \leqslant 3$.
34. $f(x, y, z) = (x + y)/(y + z)$, $\boldsymbol{\sigma}(t) = (t, \frac{2}{3}t^{3/2}, t)$; $1 \leqslant t \leqslant 2$.

★35. Show that the value of the line integral of a scalar function over a parametric curve, defined after Exercise 28, is unchanged if the curve is reparametrized.

18.2 Path Independence

The line integral of a gradient vector field depends only on the endpoints of the curve.

We saw in the last section that the line integral of a vector field along a curve from a point A to a point B depends not just on A and B, but on the path of integration itself. There is, however, an important class of vector fields for which line integrals are path independent. In this section and the next, we shall give several different ways to recognize and use such vector fields.

A vector field $\boldsymbol{\Phi}(x, y, z)$ defined on some domain D in space (or a vector field $\boldsymbol{\Phi}(x, y)$ defined on a domain in the plane) is called *conservative* if, whenever C_1 and C_2 are curves in D with the same endpoints, the line integrals $\int_{C_1}\boldsymbol{\Phi}(\mathbf{r}) \cdot d\mathbf{r}$ and $\int_{C_2}\boldsymbol{\Phi}(\mathbf{r}) \cdot d\mathbf{r}$ are equal.

Our first observation is that a vector field on D is conservative if and only if its integral around every *closed* curve in D is zero. (A conservative force field is thus one in which no net work is done, i.e., energy is conserved if a particle goes around a closed path.)

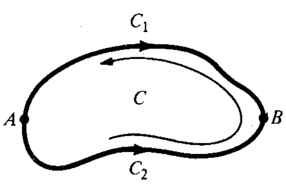

Figure 18.2.1. C_1 and C_2 have the same endpoints when $C = C_2 + (-C_1)$ is closed.

To justify this observation, we consider Figure 18.2.1, which can be interpreted in two different ways. First of all, if C_1 and C_2 are given curves from A to B, then $C = C_2 + (-C_1)$ is a closed curve (from A to A). If $\boldsymbol{\Phi}$ has the property that its integral around every closed curve is zero, then by formulas (7) and (8) in Section 18.1,

$$\int_{C_2}\boldsymbol{\Phi}(\mathbf{r}) \cdot d\mathbf{r} - \int_{C_1}\boldsymbol{\Phi}(\mathbf{r}) \cdot d\mathbf{r} = \int_{C_2+(-C_1)}\boldsymbol{\Phi}(\mathbf{r}) \cdot d\mathbf{r} = \int_C \boldsymbol{\Phi}(\mathbf{r}) \cdot d\mathbf{r} = 0;$$

so $\int_{C_1}\boldsymbol{\Phi}(\mathbf{r}) \cdot d\mathbf{r} = \int_{C_2}\boldsymbol{\Phi}(\mathbf{r}) \cdot d\mathbf{r}$. Since this is true for all curves C_1 and C_2 with common endpoints, $\boldsymbol{\Phi}$ is conservative.

The second way to look at Figure 18.2.1 is to consider the closed curve C as given; the pieces C_1 and C_2 are then manufactured by choosing points A and B on C so that $C = C_1 + (-C_2)$. Now if $\boldsymbol{\Phi}$ is conservative, then

$$0 = \int_{C_2}\boldsymbol{\Phi}(\mathbf{r}) \cdot d\mathbf{r} - \int_{C_1}\boldsymbol{\Phi}(\mathbf{r}) \cdot d\mathbf{r} = \int_{C_2+(-C_1)}\boldsymbol{\Phi}(\mathbf{r}) \cdot d\mathbf{r} = \int_C \boldsymbol{\Phi}(\mathbf{r}) \cdot d\mathbf{r};$$

so the integral of $\boldsymbol{\Phi}$ around a closed curve C is zero.

The argument just given to connect path independence with integrals around closed curves has many applications in mathematics. A related geometric example is given in Exercise 37.