8/14/2013

Applied Complex Variables (복소변수)

Bok So Byon Su

Office hour: Thursday 15.00 – 17.00

Office : 103A

References:

1) Complex Variables and Applications, by James Ward Brown and Ruel V. Churchill, 6th, 7th, or 8th edition, 2009 (Textbook)

2) Complex Analysis with Applications, by R.A. Silverman, 1974

3) Complex Analysis, by L.V. Ahlfors, [illegible], 1979 (Fields medal 1936) (Finnish Mathematician, 4/18/1907 – 10/11/1996)

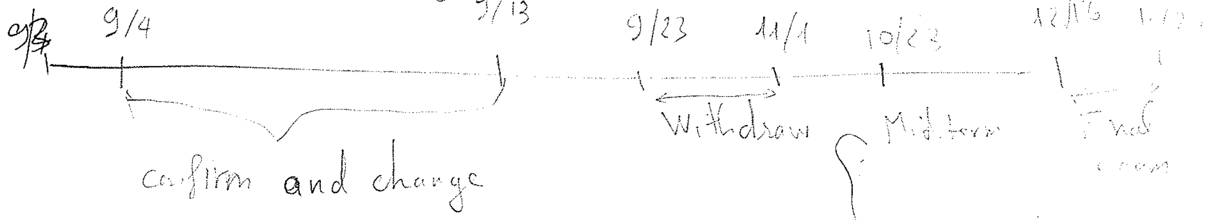
4) Complex Analysis, by Serge Lang, 4th ed, 1999. (French Mathematician, 5/19/1927 – 9/12/2005, a member of the Bourbaki group ...)

Finland

5)

Schedule:

- 9.4 ~ 13 : Confirmation and changes of course registration
- 9.23 ~ 11.1 : Withdrawal from Courses
- 10.23 : Mid term exam
- 12.16–20 : Final exam
- 9/19, 10/3 ; Holiday / days off



Motivations

$x^2+1=0 \to \exists\, i$ s.t. $i^2=-1$ $\Rightarrow z = x+iy$

i : the imaginary number $\mathbb{C} = \{z \mid (x,y) \in \mathbb{R}^2\}$

Compute improper integrals (적분 / Jeokbun)

1) $\displaystyle\int_0^{+\infty} \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$ $\quad \displaystyle\int_0^{+\infty} \frac{dx}{1+x^2+x^4} = \frac{\pi}{2\sqrt{3}}$ $\quad \displaystyle\int_0^{2\pi} \frac{d\theta}{1+a\sin\theta} = \frac{2\pi}{\sqrt{1-a^2}}$ $(|a|<1)$

2) $\displaystyle\int_0^{+\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2}$ $\quad \displaystyle\int_0^{+\infty} \frac{1-\cos x}{x^2}\,dx = \frac{\pi}{2}$

3) $\displaystyle\int_0^{+\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$ $\quad \displaystyle\int_0^{+\infty} \sin x^2\,dx = \int_0^{+\infty} \cos x^2\,dx = \frac{\sqrt{\frac{\pi}{2}}}{2}$

Compute the sum of series (급수)

1) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$; $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3} = 1.2020569031503 1 \ldots$

2) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$

3) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$

We can compute improper integrals, sum of series, by different method (applications of the residue's theorem)

$\bullet\ 1+x^4 = (1+x^2)^2 - 2x^2 = (1+x^2)^2 - (\sqrt{2}x)^2$

$= (1-\sqrt{2}x+x^2)(1+\sqrt{2}x+x^2)$

$$\frac{1}{1+x^4} = \frac{Ax+B}{1+\sqrt{2}x+x^2} + \frac{Cx+D}{1-\sqrt{2}x+x^2}$$

$A = \frac{1}{2\sqrt{2}}$

$C = -\frac{1}{2\sqrt{2}}$

$B = \frac{1}{2}$

$D = \frac{1}{2}$

$$\int_0^{+\infty} \frac{dx}{1+x^4} = \frac{1}{4\sqrt{2}} \ln \frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\Big|_0^{+\infty} + \frac{1}{2\sqrt{2}} \arctan(\sqrt{2}x+1)\Big|_0^{+\infty} + \frac{1}{2\sqrt{2}} \arctan(\sqrt{2}x-1)\Big|_0^{+\infty}$$

$$= 0 + \frac{1}{2\sqrt{2}}\left(\frac{\pi}{2}-\frac{\pi}{4}\right) + \frac{1}{2\sqrt{2}}\left(\frac{\pi}{2}+\frac{\pi}{4}\right) = \frac{\pi}{2\sqrt{2}}.$$

$\bullet\ 1+x^2+x^4 = (1+x^2)^2 - x^2 = (1+x+x^2)(1-x+x^2)$

$$\frac{1}{1+x^2+x^4} = \frac{Ax+B}{1+x+x^2} + \frac{Cx+D}{1-x+x^2} \to A = +1/2,\ C = -1/2,\ B = 1/2,\ D = 1/2$$

$$\int_0^{+\infty} \frac{dx}{1+x^2+x^4} = \frac{1}{4} \ln \frac{x^2+x+1}{x^2-x+1}\Big|_0^{+\infty} + \frac{1}{2\sqrt{3}} \arctan\left(\frac{2x+1}{\sqrt{3}}\right)\Big|_0^{+\infty} + \frac{1}{2\sqrt{3}} \arctan\left(\frac{2x-1}{\sqrt{3}}\right)\Big|_0^{+\infty}$$

$$= \frac{\pi}{2\sqrt{3}}$$

$$\bullet\ \int_0^{2\pi} \frac{d\theta}{1+a\sin\theta} = \int_0^{\pi} \frac{d\theta}{1+a\sin\theta} + \int_{\pi}^{2\pi} \frac{d\theta}{1+a\sin\theta} = \int_0^{\pi} \frac{d\theta}{1+a\sin\theta} + \int_0^{\pi} \frac{d\theta}{1-a\sin\theta}$$

$t = \theta - \pi$

$\theta = \pi + t$

let $t = \tan\frac{\theta}{2}$.

$d\theta = \frac{2\,dt}{1+t^2}$

$=$

$$\int_0^{2\pi}\frac{d\theta}{1+a\sin\theta} = \int_0^{+\infty}\frac{\frac{2\,dt}{1+t^2}}{1+a\frac{2t}{1+t^2}} + \int_0^{+\infty}\frac{\frac{2\,dt}{1+t^2}}{1-a\frac{2t}{1+t^2}}$$

$$= \int_0^{+\infty}\frac{2\,dt}{t^2+2at+1} + \int_0^{+\infty}\frac{2\,dt}{t^2-2at+1}$$

$$= \frac{2}{\sqrt{1-a^2}}\arctan\frac{t+a}{\sqrt{1-a^2}}\Big|_0^{+\infty} + \frac{2}{\sqrt{1-a^2}}\arctan\frac{t-a}{\sqrt{1-a^2}}\Big|_0^{+\infty}$$

$$= \frac{2\cdot\pi}{\sqrt{1-a^2}}$$

- $I=\int_0^{+\infty}\frac{\sin x}{x}\,dx$, $\quad I(\alpha):=\int_0^{+\infty}e^{-\alpha x}\frac{\sin x}{x}\,dx$, $\quad \alpha\geq 0$

$$I'(\alpha) = -\int_0^{+\infty}e^{-\alpha x}\sin x\,dx \overset{\text{"by exercise"}}{=} \frac{1}{1+\alpha^2}, \quad I(+\infty)=0$$

$$I(\alpha) = \arctan\alpha + \frac{\pi}{2}$$

$$\downarrow \alpha\to 0^+ \qquad \downarrow$$

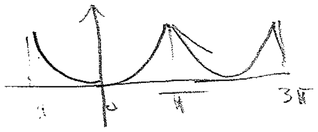
$$I = 0 + \frac{\pi}{2} = \frac{\pi}{2}.$$

- $\int_0^{+\infty}e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$, $\quad J=\int_{-\infty}^{+\infty}e^{-x^2}dx$, $\quad J^2=\int_{-\infty}^{+\infty}e^{-x^2}dx\int_{-\infty}^{+\infty}e^{-y^2}dy$

$$J^2 = \iint_{\mathbb{R}^2}e^{-(x^2+y^2)}dx\,dy \overset{\substack{x=r\cos\varphi\\ y=r\sin\varphi}}{=} \int_0^{+\infty}dr\int_0^{2\pi}e^{-r^2}r\,d\theta = \pi\int_0^{+\infty}e^{-r^2}dr^2$$

$$= -\pi e^{-r^2}\Big|_0^{+\infty} = \pi \Rightarrow J=\sqrt{\pi}.$$

- $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$ Fourier series



$2\cos(nx) = e^{inx}+e^{-inx}$

$$x^2 = \frac{\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos(nx), \quad x\in[-\pi,\pi]$$

+ $x=0 \Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = -\frac{\pi^2}{12} \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$, $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}$

+ $\begin{cases} S=\sum_1^{\infty}\frac{1}{n^2} = \frac{1}{4}S+\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} \\ -\frac{\pi^2}{12} = \frac{1}{4}S-\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\end{cases} \Bigg| \Rightarrow S=\frac{\pi^2}{6}$ & $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}$.

$$\frac{1}{2}S = \frac{\pi^2}{12}$$

+ use the Parseval's identity. $\left(\frac{\pi^2}{3}\right)^2+2\sum\left(\frac{2(-1)^n}{n^2}\right)^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}|x^2|^2dx$

$$\Downarrow$$

$$\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}.$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$$

$$\int_0^1 x^{2k}\,dx = \frac{x^{2k+1}}{2k+1}\Big|_0^1 = \frac{1}{2k+1}$$

$$\int_0^1 x^{2k}\,dx \int_0^1 y^{2k}\,dy = \frac{1}{(2k+1)^2}$$

$$\Rightarrow \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \int_0^1\int_0^1 \sum_{k=0}^{\infty} \left[(xy)^2\right]^k dx\,dy$$

$$= \int_0^1\int_0^1 \frac{1}{1-(xy)^2}\,dx\,dy$$

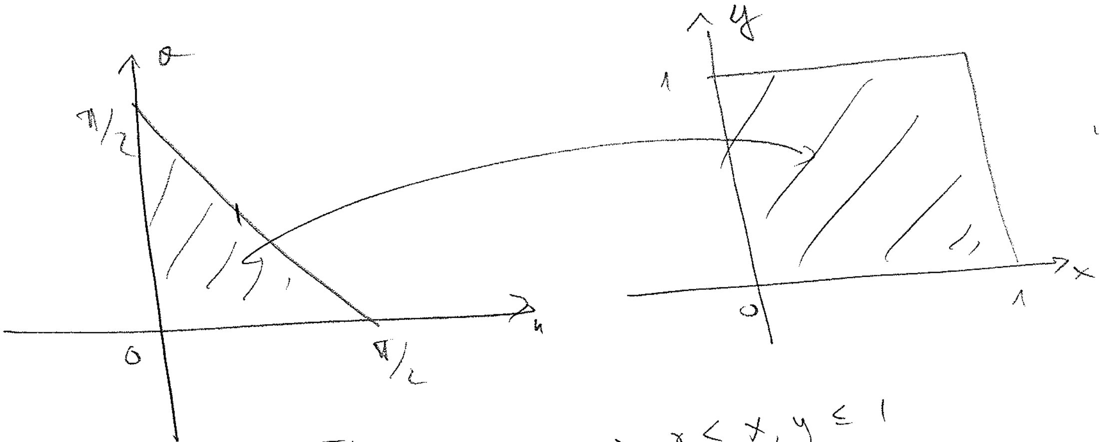
$$= \int_0^{\pi/2} du \int_0^{\pi/2-u} du = \int_0^{\pi/2} \left(\frac{\pi}{2}-u\right) du$$

$$\begin{cases} x = \dfrac{\sin u}{\cos v} \\ y = \dfrac{\sin v}{\cos u} \end{cases} \qquad = \frac{\pi}{2}\cdot\frac{\pi}{2} - \frac{1}{2}\left(\frac{\pi}{2}\right)^2 = \frac{\pi^2}{8}.$$

$$J = \begin{vmatrix} \dfrac{\cos u}{\cos v} & \dfrac{\sin u}{\cos^2 v}\sin v \\ \dfrac{\sin v \sin u}{\cos^2 u} & \dfrac{\cos v}{\cos u} \end{vmatrix}$$

$$= 1 - \frac{\sin^2 u \sin^2 v}{\cos^2 u \cos^2 v} = 1 - tg^2 u\, tg^2 v \neq 0$$

$\forall$



$$\begin{cases} 0 \le u \le \pi/2 \\ 0 \le v \le \pi/2 - u \end{cases} \longrightarrow 0 \le x, y \le 1$$

$$tg\, v = cotg\, u$$

$$v < \pi/2 - u \Rightarrow \begin{matrix} \cos v > \sin u \\ \sin v < \cos u \end{matrix}$$

$$\Rightarrow \boxed{0 < x < 1, \; 0 < y < 1.}$$

Introduction to Applied Complex Variables

$z = x + iy$: complex number

- $\mathbb{C} = \{ z = x+iy = (x,y) \mid x, y \in \mathbb{R} \}$: the complex number field
- Topology on $\mathbb{C}$: open sets, closed set, domains (open and connected), limits, distance
- $f : \mathbb{C} \supset D \to \mathbb{C}$: complex function $\left[\begin{array}{l} \lim\limits_{z \to z_0} f(z) \\ f'(z_0) := \lim \dfrac{f(z) - f(z_0)}{z - z_0} \end{array} \right.$
- f is holomorphic on D if $\exists f'(z) \; \forall z \in D$ ($f \in Hol(D)$)

☺ f is holo. $\Rightarrow$ f' is hol, ... $\exists f^{(n)}(z), \forall n$

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \cdots + \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n + \cdots$$

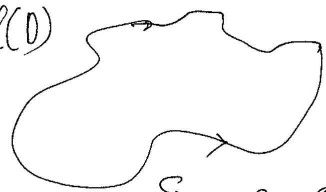
(Taylor series of f at z_0)

($f : \mathbb{R} \to \mathbb{R}$, $\exists f' \not\Rightarrow \exists f''$ (ex. $f(x) = x|x|$)

$f \in C^\infty(\mathbb{R}) \not\Rightarrow f(x) = f(x_0) + \dfrac{f'(x_0)}{1!}(x - x_0) + \cdots$

Counterexample $f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0 & x = 0 \end{cases}$ $\quad f^{(n)}(0) = 0$) $\quad e^{-\frac{1}{x^2}} = 0 \;\; \forall 0 < |x| < \varepsilon$ impossible

- Define $\int_C f(z)\,dz$
- $\oint_C f(z)\,dz = 0 \quad \forall C \subset D, f \in Hol(D)$



C

$\Downarrow$ $\qquad f : \mathbb{C} \to \mathbb{C}$

☺ $$\int_{|z - z_0| = R > 0} \frac{dz}{(z - z_0)^n} = \begin{cases} 2\pi i & \text{if } n = 1 \\ 0 & \text{if } n \neq 1 \end{cases}$$

if $f(z) := \sum\limits_{n=-\infty}^{+\infty} c_n (z - z_0)^n$ (Laurent series) $c_{-1} = \operatorname{Res}_{z_0} f(z)$

$$\int_{|z - z_0| = R} f(z)\,dz = \sum_{n=-\infty}^{+\infty} c_n \int_{|z-z_0| = R} (z - z_0)^n\,dz = 2\pi i\, c_{-1}.$$

$$= 2\pi i \operatorname{Res}_{z_0} f(z)$$

☺ $$\int_{\partial D} f(z)\,dz = 2\pi i \sum_{j=1}^{n} \operatorname{Res}_{z = z_j} f(z)$$

Real Analysis

- $\mathbb{R}$: real number field or the field of real numbers

- Real functions

$f : \mathbb{R} \supset A \longrightarrow \mathbb{R}$

$y = f(x)$

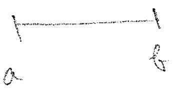
— continuous

Ex. $y = e^x, \sin x, \cos x, \frac{P(x)}{Q(x)}, \log x$

- $\lim_{x \to a} f(x) = b$

- $f'(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

- $\int_a^b f(x)\,dx$



- Real-analytic functions

$f : (a,b) \longrightarrow \mathbb{R}$

$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n, a_n \in \mathbb{R}$

Ex: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \forall x \in \mathbb{R}$

Note:

- Real-analytic $\Longrightarrow$ C^∞-smooth ($\exists f^{(n)} \ \forall n = 0,1,2,\dots$)

$\not\Longleftarrow$

Counter-example

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$f \in C^\infty(\mathbb{R})$ but f is not real-analytic

$(f^{(n)}(0) = 0 \ \forall n)$

$0 \neq f(x) \neq \sum_{n=0} \frac{f^{(n)}(0)}{n!} x^n = 0$

Complex Analysis

- $\mathbb{R}^2 \equiv \mathbb{C} = \{ z = x + iy \mid x, y \in \mathbb{R} \}$

: complex number field

Complex functions

$f : \mathbb{C} \supset D \longrightarrow \mathbb{C}$

$w = f(z), \quad z = x + iy$

$= f(x,y)$

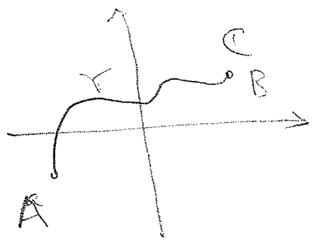
— continuous

Ex: $w = e^z, \sin z, \cos z, \frac{P(z)}{Q(z)}, \log z$

- $\lim_{z \to z_0} f(z) = w_0$

- $f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$

- $\int_\gamma f(z)\,dz$



- Analytic functions (= holomorphic f'tns)

$f : \mathbb{C} \supset D \longrightarrow \mathbb{C}$

$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n, c_n \in \mathbb{C}$

$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad z \in \mathbb{C}$

- $\exists f'(z) \ \forall z \in D \Longrightarrow$ holomorphic

$\Longrightarrow C^\infty$-smooth ($\exists f^{(n)}(z) \ \forall n$)

- Applied Complex Analysis
- main object = "complex function"

$$f: \mathbb{C} \supset D \to \mathbb{C}, \quad w = f(z).$$

- Contents

Chapter 1. Complex numbers

- complex numbers, properties, modulus, distance, complex conjugates,
- Exponential form, arguments, roots of complex numbers, Regions (domains)

($\mathbb{C}$: complex number field, topology ...)

Chapter 2. Analytic functions

· functions of a complex variable, mappings, Limits, continuity, Derivatives, Cauchy - Riemann equations, Analytic functions, Harmonic functions, Reflection Principle,

Chapter 3. Elementary functions

The exponent function, $\log z$, trigonometric functions ($\cos z, \sin z, \ldots, \tan z, \cot z$)

Chapter 4. (Contour) Integrals

. Contour Integrals, Anti-derivatives, Cauchy - Goursat theorem, Simply (Multiply) connected domains, Cauchy Integral formula, Liouville's theorem, Maximum Modulus principle

Chapter 5. Series

. Convergence of sequences (series), Taylor series, Laurent series
. Absolute and Uniform Convergence of Power series
. Integration and Differentiation of Power series
. Uniqueness of series representations
. Multiplication and Division of Power series
. Analytic Continuation.

Chapter 6. Residues and Poles

- Isolated singular points (removable, poles, essential pts)
- Residues, Residue theorem, zeros and poles of analytic functions
- Behaviour of functions near isolated singular points.

Chapter 7. Applications of Residues

- Evaluation of improper integrals (involving sines & cosines)
- Indented paths, Integration along a Branch cut
- Argument Principle and Rouché's theorem
- Inverse Laplace transforms.

1 장 / Jang)

Chapter 1 . Complex numbers (복소수 / Bok sô su)

(review)

1. Complex numbers
2. Basic (algebraic) properties
3. Moduli / mɒdʒələs /
4. Complex conjugates
5. Exponential form
6. Roots of complex numbers /ɹu:t/
7. Regions in the complex plane (ri:dʒən)

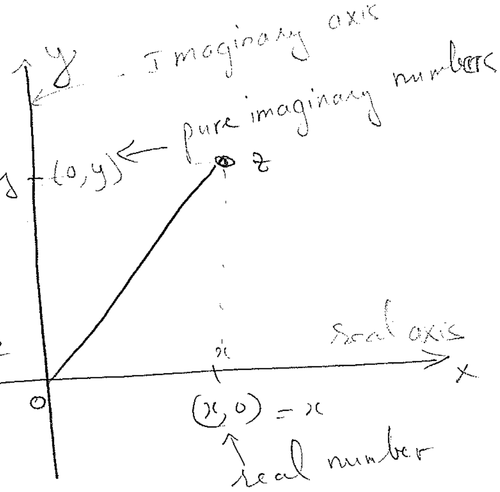
1. Complex numbers

We know that the equation $x^2+1=0$ has no solution $x \in \mathbb{R}$.
(admit)
Suppose that $x^2+1=0$ has a solution, i.e. $\exists\, i$ satisfying $i^2+1=0 \Leftrightarrow i^2=-1$. Then we have the following

- $z^2+1=0 \Rightarrow z = \pm i$
- $P_n(z) = a_n z^n + \dots + a_1 z + a_0 = 0$ $(a_n \neq 0)$ has n solutions $z_1, \dots, z_n \in \mathbb{C}$ (Fundamental Theorem of Algebra)
- $P_n(x) = (x - x_1) \dots$ [illegible]
  $x_j \in \mathbb{R}$

....

2. Basic algebraic properties



$$z = (x, y) = x(1, 0) + y(0, 1) = x + iy$$

$\mathbb{C} = \{z = x + iy \mid x, y \in \mathbb{R}\} = \mathbb{R}^2$

$x = \operatorname{Re} z$ (real part of z)

$y = \operatorname{Im} z$ (imaginary part of z)

Note: $z_1 = z_2 \Leftrightarrow \begin{cases} \operatorname{Re} z_1 = \operatorname{Re} z_2 \\ \operatorname{Im} z_1 = \operatorname{Im} z_2 \end{cases} \Leftrightarrow$

$\Leftrightarrow$ z_1, z_2 correspond to the same point in the complex plane ($\mathbb{C}$)

Define

a) Sums and products

For complex numbers $z_1 = (x_1, y_1) = x_1 + iy_1$
$z_2 = (x_2, y_2) = x_2 + iy_2$

Def.

$$z_1 + z_2 := (x_1 + x_2,\ y_1 + y_2)$$

$$z_1 \cdot z_2 := (x_1 x_2 - y_1 y_2,\ x_1 y_2 + x_2 y_1) \quad (\Rightarrow i^2 = -1)$$

b) Properties

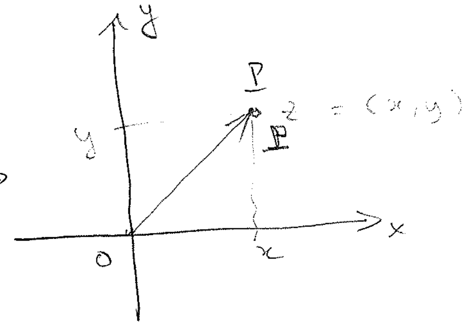
- $z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$
- $\dfrac{z_1}{z_2} = \dfrac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\,\dfrac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2} \quad (z_2 \neq 0)$
- $z^{-1} = \dfrac{x}{x^2+y^2} + i\,\dfrac{-y}{x^2+y^2} = \dfrac{1}{z}$
- $(z_1 z_2)^{-1} = z_1^{-1} z_2^{-1}$
- $\dfrac{z_1}{z_2}\cdot\dfrac{z_3}{z_4} = \dfrac{z_1 z_3}{z_2 z_4}$
- $(z_1+z_2)^n \underset{\text{(exercise)}}{=} \displaystyle\sum_{k=0}^{n} \binom{n}{k} z_1^k z_2^{n-k} \quad (n = 1, 2, \dots)$

  (Binomial formula) $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$ binomial coefficients (choose k of n)

$(\mathbb{C}, +, \cdot)$ is a field, it is called the complex number field.
$(\mathbb{R}, +, \cdot)$ $\quad \mathbb{R}$ is the real number field

3. Moduli, vectors, distance.

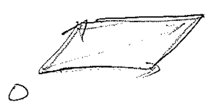
a) Vectors

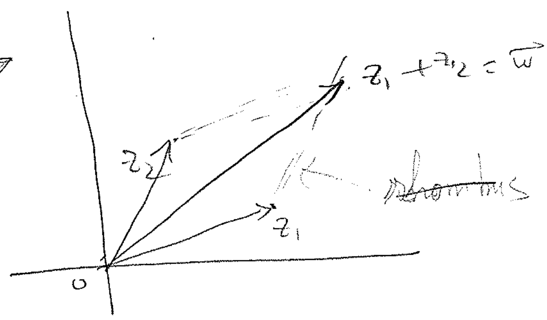


$z = x + iy \overset{1-1}{\Longleftrightarrow} \vec{z} = (x, y) = \overrightarrow{OP}$

$z_1 + z_2 \quad \Longleftrightarrow \vec{z_1} + \vec{z_2} = \vec{w}$

(by parallelogram)





b) Modulus (or absolute value)

For $z = x + iy$

$|z| := \sqrt{x^2+y^2} = |\overrightarrow{OP}|$ = the distance between (x,y) & O.

c) Distance

$d(z_1, z_2) := |z_1 - z_2|$

Check
1) $d(z_1,z_2) \geq 0$, $d(z_1,z_2) = \dots$ iff $z_1 = z_2$
2) $d(z_1,z_2) = d(z_2,z_1)$
3) $d(z_1,z_2) \leq d(z_1,z_3) + d(z_3,z_2)$

d) Triangle inequality

$$|z_1+z_2| \le |z_1|+|z_2|$$

Proof (later).

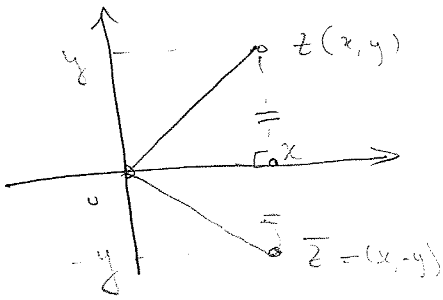
$$|z_1+z_2|^2 = (z_1+z_2)\overline{(z_1+z_2)} = (z_1+z_2)(\bar z_1+\bar z_2)$$
$$= |z_1|^2+|z_2|^2 + z_1\bar z_2 + z_2\bar z_1$$
$$= |z_1|^2 + 2\,\mathrm{Re}(z_1\bar z_2) + |z_2|^2$$
$$\le |z_1|^2 + 2|z_1\bar z_2| + |z_2|^2$$
$$\le |z_1|^2 + 2|z_1||z_2| + |z_2|^2$$
$$= (|z_1|+|z_2|)^2 \quad \square$$

9/3/2013

4. Complex conjugates

For $z = x+iy$, $\bar z := x-iy$

$\bar z$ is the reflection in the real axis of z.



<u>Properties</u>

① $\overline{z_1 \pm z_2} = \bar z_1 \pm \bar z_2$, $\overline{z_1 z_2} = \bar z_1 \bar z_2$, $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar z_1}{\bar z_2}$ $(z_2 \ne 0)$
(Ex)

② $\mathrm{Re}\, z = \frac{z+\bar z}{2}$, $\mathrm{Im}\, z = \frac{z-\bar z}{2i}$

③ $|z|^2 = z\bar z$ (ex.) $\left(\frac{z_1}{z_2} = \frac{z_1\bar z_2}{|z_2|^2} = \cdots \right)$

④ $|z_1 z_2| = |z_1||z_2|$, $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

(proof · $|z_1z_2|^2 = z_1z_2\overline{z_1z_2} = z_1\bar z_1 z_2\bar z_2 = |z_1|^2|z_2|^2 = (|z_1||z_2|)^2$

· $\left|\frac{z_1}{z_2}\right|^2 = \frac{z_1}{z_2}\cdot\overline{\left(\frac{z_1}{z_2}\right)} = \frac{|z_1|^2}{|z_2|^2} = \left(\frac{|z_1|}{|z_2|}\right)^2$ $\square$)

/a·gjŭ mənt/ /ɛkspə'nɛnʃəl/

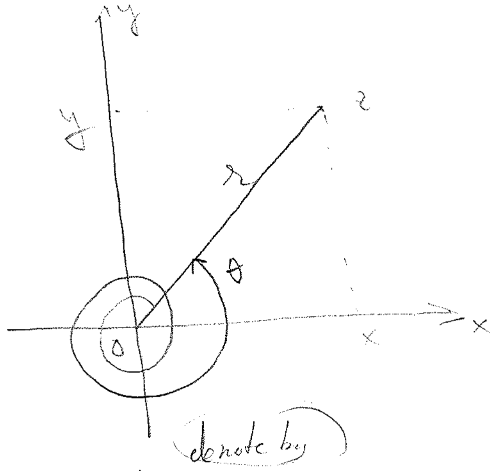
5. Argument, Polar form, Exponential form

a) Argument

$z = x + iy \neq 0$

$r := |z| > 0$

·) $\exists\, \theta \in \mathbb{R}$ such that



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

θ is called an <u>argument</u> of z, $\theta \overset{\text{denote by}}{=:} \arg z$

·) $\exists!\ \theta_0 \in [-\pi, \pi]$ such that $\begin{cases} x = r\cos\theta_0 \\ y = r\sin\theta_0 \end{cases} \rightsquigarrow \operatorname{tg}\theta_0 = y/x$

θ_0 is called the principal value of $\arg z$,

$\theta_0 \overset{\text{denote by}}{=:} \operatorname{Arg} z \qquad -\pi < \theta_0 \le \pi$

<u>Note that:</u>

$$\arg z = \operatorname{Arg} z + 2n\pi \ (n = 0, \pm 1, \pm 2, \dots) \ (n \in \mathbb{Z}).$$

Example

1) $\operatorname{Arg}(1+i) = \frac{\pi}{4}$, $\arg(1+i) = \frac{\pi}{4} + 2n\pi$, $n \in \mathbb{Z}$

2) $\operatorname{Arg}\left(\frac{i}{-2-2i}\right) =$, $\arg\left(\frac{i}{-2-2i}\right) =$

· <u>Properties</u>

+ $\arg(z_1 z_2) = \arg z_1 + \arg z_2$

+ $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2 \qquad (z_1, z_2 \neq 0)$.

b) Polar form

$z = x + iy \neq 0$

$r = |z|$
$\theta = \arg z$
$\Rightarrow \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$\Rightarrow \quad z = r(\cos\theta + i\sin\theta)$: polar form of z

Properties

$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

+ $z_1 z_2 = r_1 r_2 (\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$

+ $\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)))$ $\quad r_2 \neq 0$

+ $z^n = r^n(\cos(n\theta) + i\sin n\theta)$.

Proof

$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)$

$= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)$ □

Cor./Note:
$$\boxed{\begin{aligned} &\cdot \arg(z_1 z_2) = \arg z_1 + \arg z_2 \\ &\cdot \arg \frac{z_1}{z_2} = \arg z_1 - \arg z_2 \end{aligned}}$$

c) Exponential form

$$e^{i\theta} := \cos\theta + i\sin\theta \quad \text{(Euler's formula)}$$

$\Rightarrow z = re^{i\theta}$: exponential form

Properties $\quad z_1 = r_1 e^{i\theta_1}, \; z_2 = r_2 e^{i\theta_2}$

1) $e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$; $e^{i\theta_1}/e^{i\theta_2} = e^{i(\theta_1-\theta_2)}$

2) $z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$; $\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$

3) $z^n = r^n e^{in\theta}$

Note : $\underset{(r=1)}{(e^{i\theta})^n} = e^{in\theta} \Leftrightarrow \boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta}$

(de Moivre's formula)

Example $\quad n=3$

$$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$$

$\Leftrightarrow \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta = \cos 3\theta + i\sin 3\theta$

$$\Leftrightarrow \begin{cases} \cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta = 4\cos^3\theta - 3\cos\theta \\ \sin 3\theta = -\sin^3\theta + 3\cos^2\theta\sin\theta = 3\sin\theta - 4\sin^3\theta . \end{cases}$$

—

6. Roots of complex numbers $\quad (n = 2, 3 \dots)$

Def. $z \in \mathbb{C}$ is called an nth root of $z_0 = r_0 e^{i\theta_0}$ if $z^n = z_0$. $\left(z = \sqrt[n]{z_0} \right)$

Note. $z = re^{i\theta}, \quad z_0 = r_0 e^{i\theta_0}$

$$z^n = z_0 \iff (re^{i\theta})^n = r_0 e^{i\theta_0}$$

$$\iff r^n e^{in\theta} = r_0 e^{i\theta_0} \iff \begin{cases} r^n = r_0 \\ e^{in\theta} = e^{i\theta_0} \end{cases}$$

$$\iff \begin{cases} r = \sqrt[n]{r_0} \\ n\theta = \theta_0 + 2k\pi, \quad k \in \mathbb{Z} \end{cases}$$

$$\iff \begin{cases} r = \sqrt[n]{r_0} \\ \theta = \dfrac{\theta_0 + 2k\pi}{n}, \quad k \in \mathbb{Z} \end{cases} \iff \sqrt[n]{z_0} = \sqrt[n]{r_0}\, e^{i\frac{\theta_0 + 2k\pi}{n}}, \ k \in \mathbb{Z} \quad (k = 0, \dots, n-1)$$

$$c_0 = \sqrt[n]{r_0}\, e^{i\frac{\theta_0}{n}}$$

$$c_1 = \sqrt[n]{r_0}\, e^{i\frac{\theta_0 + 2\pi}{n}}$$

$$\dots$$

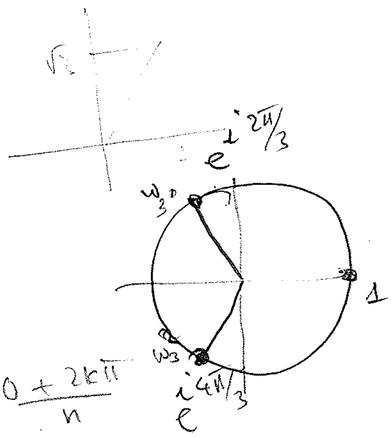
$$c_k = \sqrt[n]{r_0}\, e^{i\frac{\theta_0 + 2k\pi}{n}}$$

$$\vdots$$

$$c_{n-1} = \sqrt[n]{r_0}\, e^{i\frac{\theta_0 + 2(n-1)\pi}{n}}$$

$\sqrt[n]{z_0} = \{c_0, c_1, \dots, c_{n-1}\}$

$c_k = \left(e^{2\pi i/n}\right)^k = w_n^k$



Example.

1) $z_0 = 1 = 1e^{i \cdot 0}$, $\quad \sqrt[n]{1} = e^{i\frac{0 + 2k\pi}{n}} = e^{\frac{2k\pi i}{n}}$, $\quad k = 0, \dots, n-1$

Let $w_n := e^{i\frac{2\pi}{n}}$. $\quad \Rightarrow 1, w_n, w_n^2, \dots, w_n^{n-1}$ are nth-root of unity.

2) $\sqrt[n]{z_0} = \{c, cw_n, \dots, cw_n^{n-1}\}$, c is an nth-root of $z_0 \neq 0$

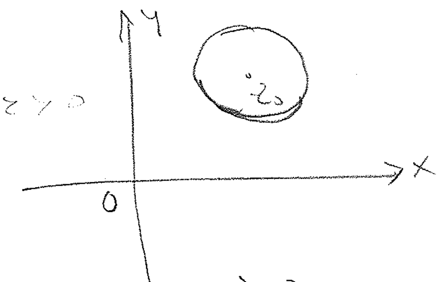
3) $\sqrt{1 + i\sqrt{3}} = \sqrt{2e^{i\frac{\pi}{3}}} = \sqrt{2}\, e^{i\left(\frac{\pi}{6} + k\pi\right)}$, $\quad k = 0, 1$

$c_0 = \sqrt{2}\, e^{i\pi/6}$, $\quad c_1 = \sqrt{2}\, e^{i\frac{5\pi}{6}}$.

7 Regions in the Complex plane

disk, disc

$B_\varepsilon(z_0) = D_\varepsilon(z_0) = \{ z \in \mathbb{C} : |z - z_0| < \varepsilon \}$, $\varepsilon > 0$



ε- neighborhood of z_0

$\overset{\circ}{B}_\varepsilon(z_0) = \overset{\circ}{D}_\varepsilon(z_0) = \{ z \in \mathbb{C} : 0 < |z - z_0| < \varepsilon \} = D_\varepsilon(z_0) \setminus \{z_0\}$

ε- deleted neighborhood of z_0.

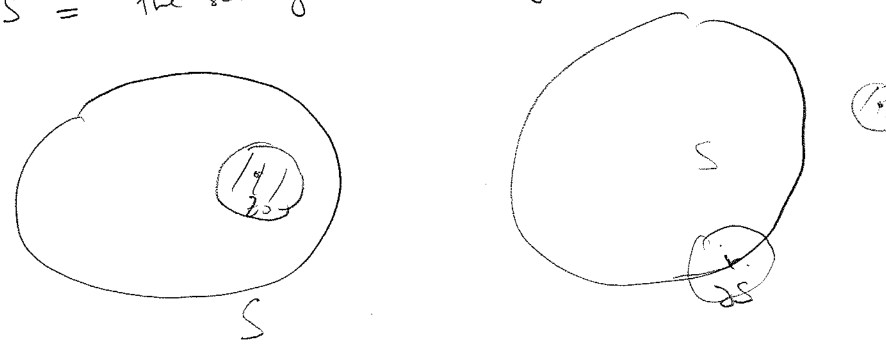
<u>Def.</u> $S \subset \mathbb{C}$.

i) z_0 is said to be <u>an interior</u> point of S if $\exists\, \varepsilon > 0$ s.t. $D_\varepsilon(z_0) \subset S$. // $\operatorname{int} S = \overset{\circ}{S}$ = the set of all interior pts of S

ii) z_0 is said to be <u>an exterior</u> point of S if $\exists\, \varepsilon > 0$ s.t. $D_\varepsilon(z_0) \subset \mathbb{C} \setminus S$.

iii) z_0 is a boundary point of S if $\forall\, \varepsilon > 0$ $D_\varepsilon(z_0) \cap S \neq \phi$ and $D_\varepsilon(z_0) \cap (\mathbb{C} \setminus S) \neq \phi$.

∂S = the set of all boundary points of S.



<u>Example</u>

1) $S = \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$

$\overset{\circ}{S} = S$, $\partial S = \{ |z| = 1 \}$

2) $S = \{ z \in \mathbb{C} : 0 < |z| \leq 1 \}$

$\overset{\circ}{S} = \overset{\circ}{\mathbb{D}}$, $\partial S = \{ |z| = 1 \} \cup \{0\}$

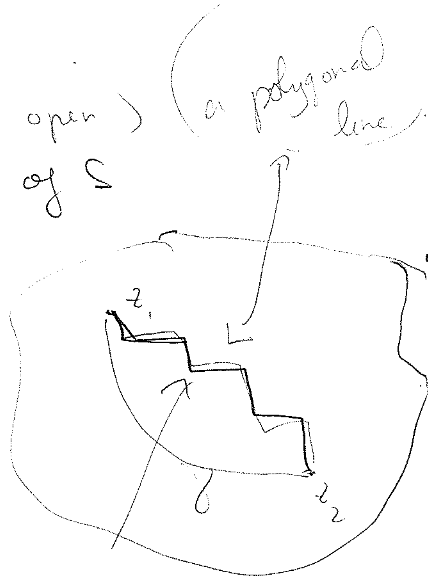
3) $S = \mathbb{D} \cup [1, 2]$, $\overset{\circ}{S} = \mathbb{D}$, $\partial S = \partial \mathbb{D} \cup [1, 2]$

Def

i) $S \subset \mathbb{C}$ is open if $\mathring{S} = S$

ii) $S \subset \mathbb{C}$ is closed if $\partial S \subset S$ (or $\mathbb{C} \setminus S$ is open)

iii) $\overline{S} = S \cup \partial S$: the closure of S



Def S is connected if

$\forall z_1, z_2 \in S$, $\exists\ \gamma : [0,1] \to S$ is continuous s.t. $\gamma(0) = z_1$, $\gamma(1) = z_2$

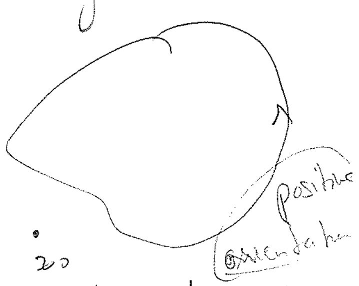
$\Leftrightarrow$ $\exists$ a polygonal line L, consisting of finite number of line segments joined end to end, $L \subset S$.

finite number of line segments.

(영역)

Def i) $D \subset \mathbb{C}$ is called a domain if

$\begin{cases} D \text{ is open}, & D \neq \emptyset \\ D \text{ is connected.} \end{cases}$



Ex $D = \{|z| < 1\}$

ii) Region = domain $\cup$ (some, none or all) of boundary points of this domain.

$\Omega_1 = \{0 < |z| \leq 1\}$, $\Omega_2 = \{|z| \leq 1\}$.

Def. S is bounded if $\exists R > 0$ s.t. $S \subset \{z \mid |z| < R\} = D_R(0)$

Ex

1) $S = \{0 < |z| \leq 1\}$ bounded region

2) $S = \{z \in \mathbb{C} \mid \operatorname{Im} z \geq 0\}$: unbounded region

Def z_0 is an accumulation point of S if $D_\varepsilon(z_0) \cap S \neq \emptyset$ $\forall \varepsilon > 0$

S' = the set of ______ S ______

Ex 0 is the only accumulation point of $S = \{\frac{1}{n} \mid n = 1, 2, \cdots\}$

Note: S is closed $\Leftrightarrow$ S contains all of its accumulation points

Proof. Exercise

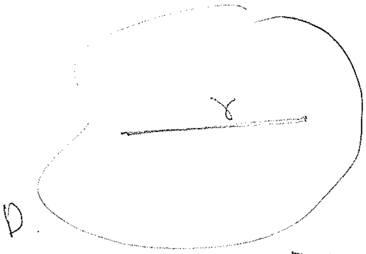
$\oplus$ $F(z) = f(z)$ $\forall z \in D$.

For $z = x + i0 = x \in \gamma$

$$F(x) = F(x+i0) = \overline{f(\overline{x+i0})} = \overline{f(x)} \overset{\text{by assumption}}{=} f(x)$$

Thus $F(z) = f(z)$ $\forall z = x \in \gamma$.

By Lemma, $F(z) = f(z)$
$\forall z \in D$.



"$\Rightarrow$" Suppose that $\overline{f(\bar z)} = f(z)$ $\forall z \in D$.

Then $\overline{f(x)} = f(x) \Rightarrow f(x) \in \mathbb{R}$ $\forall x \in \gamma$. $\boxed{X}$

Example

1) $f(z) = z^2 + 2013 \in Hol(\mathbb{C})$, $\gamma = \mathbb{R}$

$f(x) = x^2 + 2013 \in \mathbb{R}$ $\forall x \in \mathbb{R}$

$\overline{f(\bar z)} = \overline{\bar z^2 + 2013} = z^2 + 2013 = f(z)$

2) $f(z) = z + i$

- $\overline{f(\bar z)} = \overline{\bar z + i} = z - i \neq f(z)$
- $f(x) = x + i \notin \mathbb{R}$.

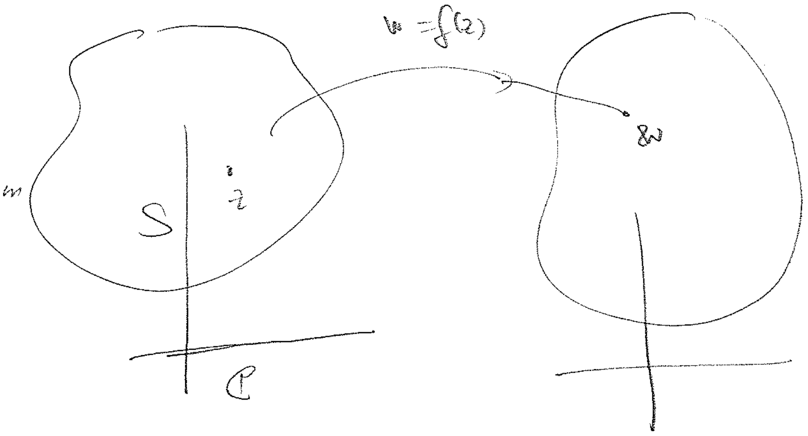
Chapter 2. Analytic functions

(해석 함수 / He Seok ham su)

1. Functions of a complex variable
2. Mappings
3. Limits
4. Continuity
5. Derivatives
6. Differentiation formulas
7. Cauchy - Riemann equations
8. Analytic functions
9. Harmonic functions
10. Uniquely determined analytic functions
11. Reflection Principle

1 Functions of a complex variable

<u>Def.</u> A function is: a map $f: \mathbb{C} \supset S \to \mathbb{C}$, i.e.,



$S \ni z \longmapsto w = f(z)$

S: the domain of definition of f.

(예제 / Ye Jae)

<u>Ex.</u>

1) $f = z^2 : \mathbb{C} \to \mathbb{C}$

2) $f = \frac{1}{z} : \mathbb{C}^* = \mathbb{C} \setminus \{0\} \to \mathbb{C}$

We can write $f(z) = u(z) + i v(z)$ $\qquad u = \mathrm{Re}(f)$

$= u(x,y) + i v(x,y)$ $\qquad v = \mathrm{Im}(f)$

$u : \mathbb{R}^2 \supset S \longrightarrow \mathbb{R}$ is the real part of f

$v : \mathbb{R}^2 \supset S \longrightarrow \mathbb{R}$ is the imaginary part of f.

<u>Ex</u>

1) $f = z^2 = (x+iy)^2 = \underbrace{x^2 - y^2}_{u} + \underbrace{2xy}_{v}\, i$ / $f = z^n$

2) $f = \frac{1}{z} = \frac{1}{x+iy} = \underbrace{\frac{x}{x^2+y^2}}_{u} - \underbrace{\frac{y}{x^2+y^2}}_{v}\, i$

3) $f(z) = |z|^2$

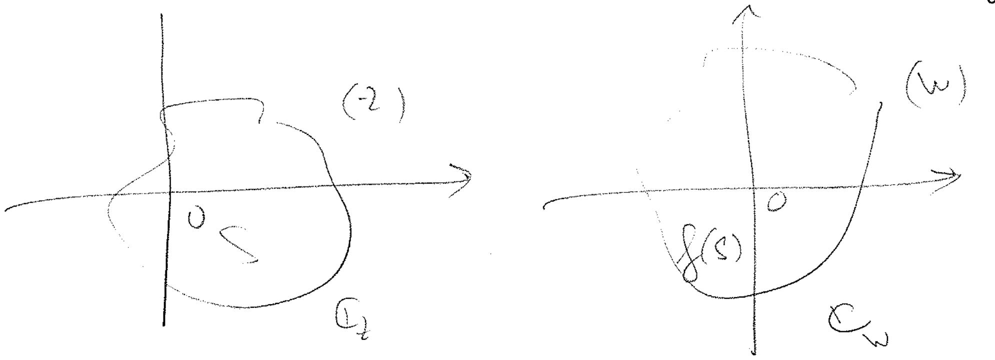
4) $f(z) = z^{1/2} = \sqrt{r}\, e^{i\frac{\Theta_0}{2}}$, $-\pi < \theta \leq \pi$

$z = r e^{i\theta_0}$, $\theta_0 = \mathrm{Arg}\, z$

2 Mappings

Given $w = f(z), z \in S$, we cannot draw the graph of f..

But we can draw the z-plane and w-plane separately

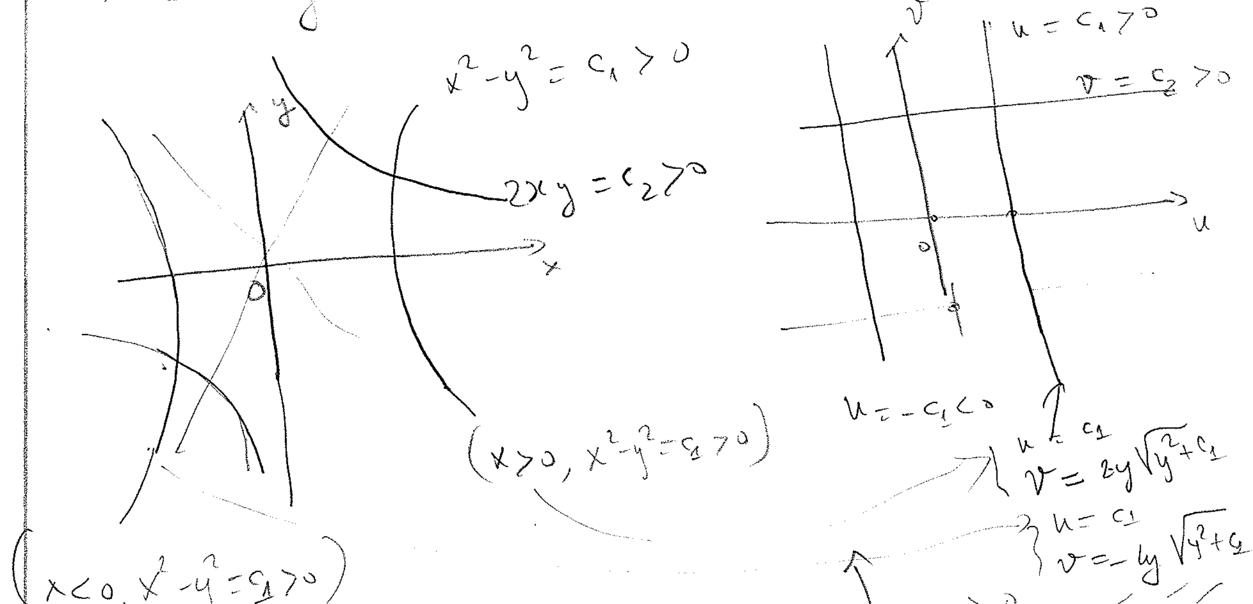


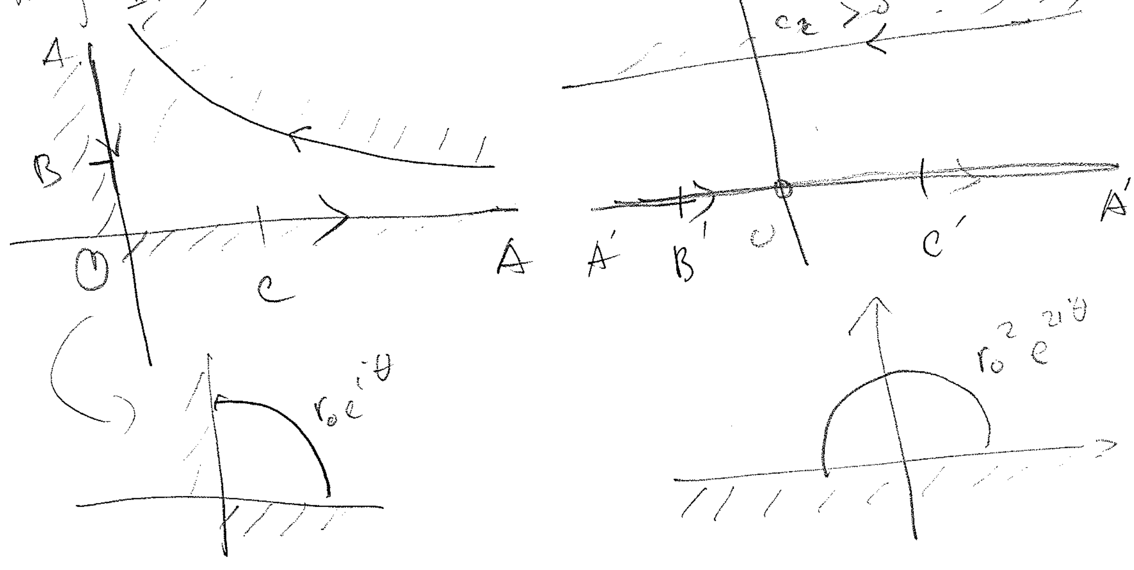
Example

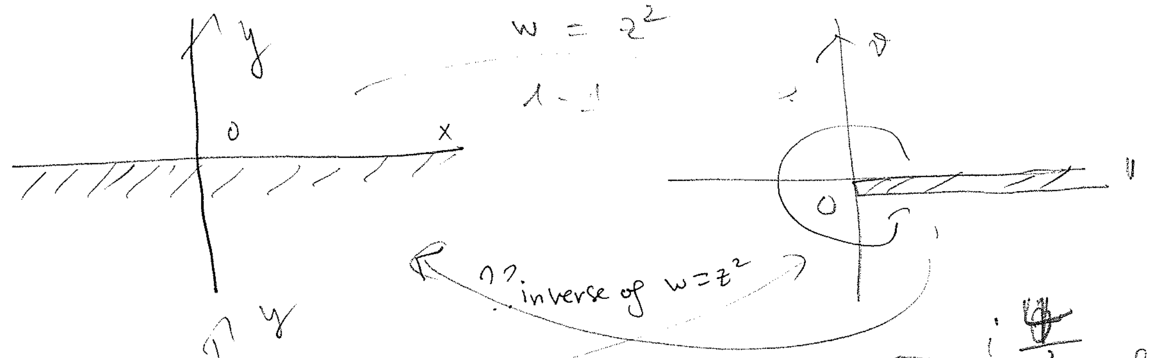
1) $w = z^2 = (x+iy)^2 = x^2 - y^2 + 2xy\,i$

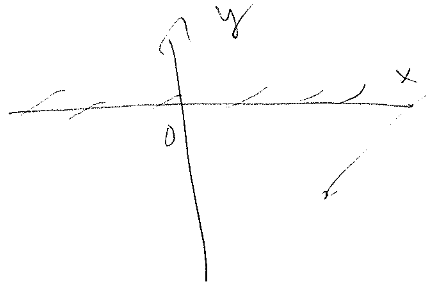
$u = x^2 - y^2$

$v = 2xy$









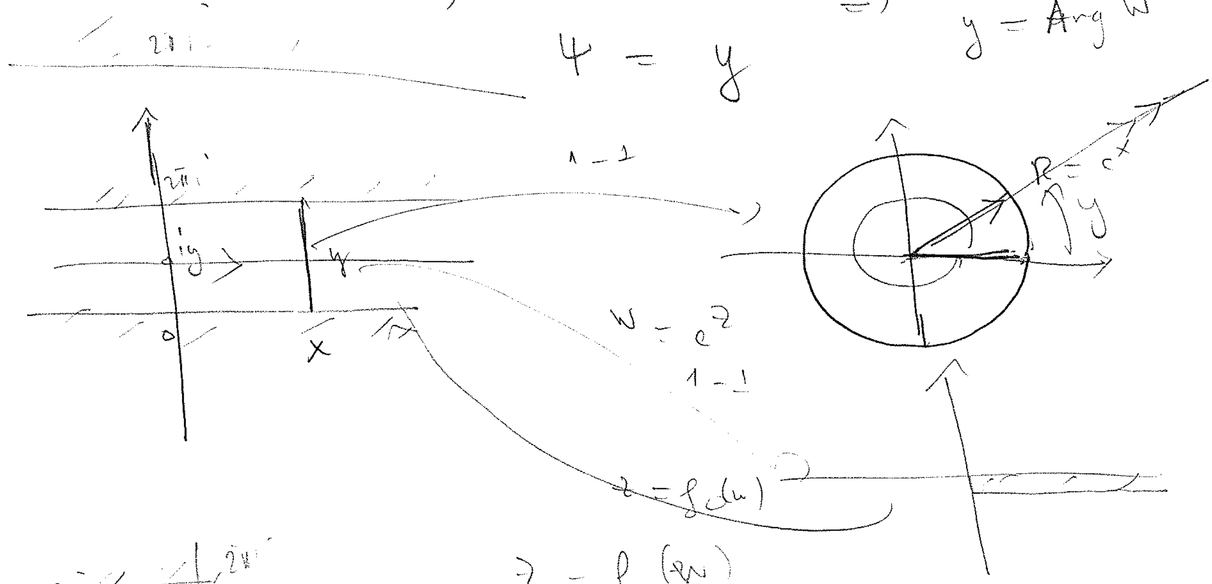
$$z = \sqrt{w} = \sqrt{R}\, e^{i\frac{\varphi}{2}} = f_+(w)$$

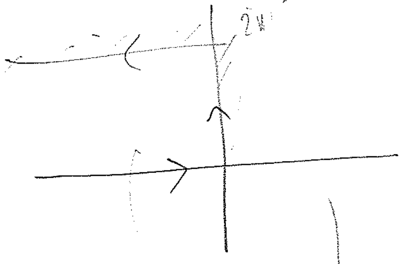
$$0 < \varphi < 2\pi$$

$$z = -\sqrt{w} = -\sqrt{R}\, e^{i\frac{\varphi}{2}} = f_-(w)$$

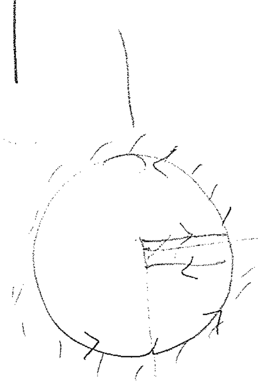
2) $w = e^z := e^{x+iy} := e^x e^{iy} = e^x(\cos y + i \sin y)$

$= R e^{i\varphi}$, $\quad R = e^x$, $\varphi = y$ $\Rightarrow$ $x = \log|w|$, $y = \operatorname{Arg} w$





$$z = f_0(w) = \log|w| + i \operatorname{Arg} w$$



$$z = f_1(w) = \log|w| + i(\varphi + 2\pi i)$$

$$z = f_k(w) = \log|w| + i(k + 2\pi i$$

3. Limits (극한 / Guk han) $z_0 \in S'$

$f : \mathbb{C} \supset S \longrightarrow \mathbb{C}$, z_0 is an accumulation pt of S.

$$\lim_{z \to z_0} f(z) = w_0 \overset{def}{\Longleftrightarrow} \forall \varepsilon > 0 \ \exists \delta > 0 \text{ s.t. } |f(z) - w_0| < \varepsilon \text{ whenever } \begin{cases} 0 < |z - z_0| < \delta \\ z \in S \end{cases}$$

$$\Longleftrightarrow \forall \{z_n\} \in S, \ z_n \to z_0 \text{ as } n \to \infty \Longrightarrow f(z_n) \to w_0 \text{ as } n \to \infty$$

Note : if $\exists \lim_{z \to z_0} f(z)$ then it is unique, i.e.,

$$\left.\begin{aligned} \lim_{z \to z_0} f(z) &= w_0 \\ \lim_{z \to z_0} f(z) &= w_1 \end{aligned}\right\} \Longrightarrow w_1 = w_0$$

Proof (Exercise) Suppose that $w_1 \neq 0$

$$\varepsilon = \frac{1}{2} |w_0 - w_1| > 0$$

$\lim_{z \to z_0} f(z) = w_0$

$\Rightarrow \exists \delta_1 > 0 : \ |f(z) - w_0| < \varepsilon \ \forall 0 < |z - z_0| < \delta_1$

$\lim_{z \to z_0} f(z) = w_1$

$\Rightarrow \exists \delta_2 > 0 : \ |f(z) - w_1| < \varepsilon$ ——— δ_2

$\delta := \min\{\delta_1, \delta_2\} > 0$. Then Fix z s.t. $0 < |z - z_0| < \delta$

$$\Rightarrow 0 < 2\varepsilon = |w_0 - w_1| \le |w_0 - f(z)| + |w_1 - f(z)| < \varepsilon + \varepsilon = 2\varepsilon$$

it is a contradiction. So $w_1 = w_0$ ☒.

Example 1) $\nexists \lim_{z \to 0} \frac{\bar{z}}{z}$

2) $\lim_{z \to i} z^2 = -1$.

(정리 / Jeong li) $f : \mathbb{C} \supset S \to \mathbb{C}$, $z_0 \in S'$

Theorem 1. Suppose that $f(z) = u(x,y) + iv(x,y)$ and $z_0 = x_0 + iy_0$, $w_0 = u_0 + iv_0$. Then

$$\lim_{z \to z_0} f(z) = w_0 \iff \begin{cases} \lim_{(x,y) \to (x_0,y_0)} u(x,y) = u_0 \\ \lim_{(x,y) \to (x_0,y_0)} v(x,y) = v_0 \end{cases}$$

Proof (Exercise) $\max\{|x|,|y|\} \le |x + iy| = \sqrt{x^2+y^2} \le |x| + |y|$

$$|f(z) - w_0| = |u(x,y) - u_0 + i(v(x,y) - v_0)|$$

$$\max\{|u - u_0|, |v - v_0|\} \le |f(z) - w_0| \le \underbrace{|u-u_0|}_{<\varepsilon/2} + \underbrace{|v-v_0|}_{<\varepsilon/2}$$

"$\Longrightarrow$" $\lim_{z \to z_0} f(z) = w_0 \Rightarrow \forall \varepsilon > 0\ \exists \delta > 0$ s.t.

$|f(z) - w_0| < \varepsilon$ whenever $0 < |z - z_0| < \delta$

$\Updownarrow$

$0 < |(x,y) - (x_0,y_0)| < \delta$

Since $\max\{|u-u_0|, |v-v_0|\} \le |f(z) - w_0| < \varepsilon$

$\Rightarrow \lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ & $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$

"$\Longleftarrow$" Suppose that $\begin{cases} u \to u_0 \\ v \to v_0 \end{cases}$ as $(x,y) \to (x_0,y_0)$

$\Rightarrow \forall \varepsilon > 0\ \exists \delta_1 > 0$ s.t. $|u - u_0| < \varepsilon/2$ whenever $|(x,y) - (x_0,y_0)| < \delta_1$

$\exists \delta_2 > 0$ ——— $|v - v_0| < \varepsilon/2$ ——— δ_2

Let $\delta = \min\{\delta_1, \delta_2\} > 0$.

Then $|f(z) - w_0| \le |u - u_0| + |v - v_0| < \frac{\varepsilon}{2} + \varepsilon/2 = \varepsilon$ whenever ——— $< \delta$

$\Rightarrow \lim_{z \to z_0} f(z) = w_0$ ☒

Theorem 2. Suppose that $\lim_{z\to z_0} f(z) = \alpha_0$.

$\lim_{z\to z_0} g(z) = \beta_0$

Then

1) $\lim_{z\to z_0} [f(z) \pm g(z)] = \alpha_0 \pm \beta_0$

2) $\lim_{z\to z_0} f(z) \cdot g(z) = \alpha_0 \beta_0$

3) $\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{\alpha_0}{\beta_0}$ if $\beta_0 \neq 0$

Proof (Exercise)

use 1) $|f \cdot g - \alpha_0\beta_0| = |f\cdot g - \alpha_0 \cdot g + \alpha_0 g - \alpha_0\beta_0|$

$= |(f - \alpha_0)\cdot g + \alpha_0 (g - \beta_0)|$

$\leq \frac{\varepsilon}{2M} \cdot M + M' \cdot \frac{\varepsilon}{2M'}$

$\leq \varepsilon \qquad M, M' > 0 \qquad \boxtimes$

use 2) $\left|\frac{1}{g} - \frac{1}{\beta_0}\right| = \left|\frac{g - \beta_0}{g \cdot \beta_0}\right| \leq \frac{|g - \beta_0|}{\frac{\beta_0^2}{2}}$.

$|g| > \frac{\beta_0}{2}$ $\qquad \boxtimes$.

Cor

Ex

1) $\lim_{z\to z_0} c = c_0$, $\lim_{z\to z_0} z = z_0$, $\lim_{z\to z_0} z^n = z_0^n$

2) $\lim_{z\to z_0} P(z) = P(z_0)$. (by Induction)

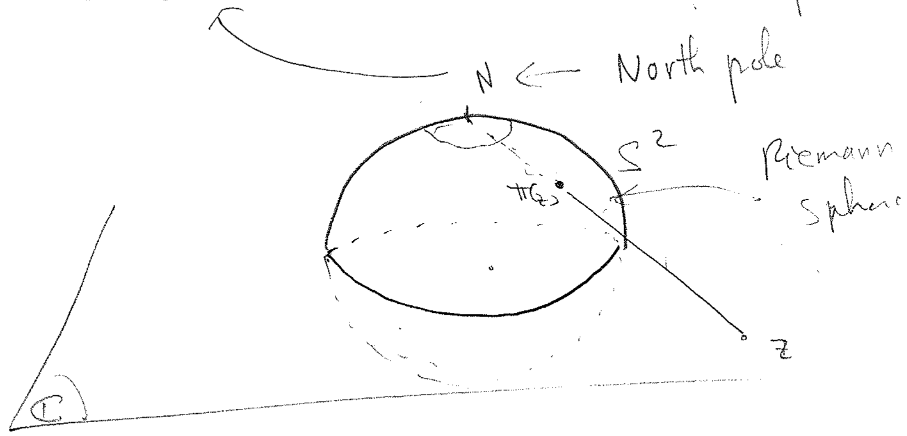
for any polynomial $P(z) = a_n z^n + \dots + a_1 z + a_0$.

$(a_j \in \mathbb{C})$

$n \geq 0$

- limits involving the point at infinity

$\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$: the extended complex plane



$\pi : \mathbb{C} \xrightarrow{1-1} S^2 \setminus \{N\}$: homeomorphism π, π^{-1} are continuous

$\quad z \longmapsto \pi(z)$ $\mathbb{C} \cong S^2 \setminus \{N\}$

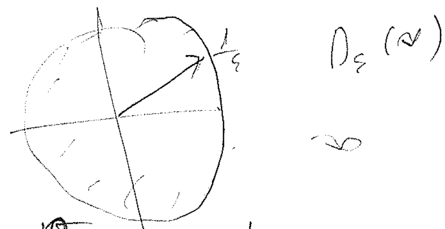
"the stereographic projection"

$\pi : \mathbb{C} \cup \{\infty\} \xrightarrow{1-1} S^2$: homeomorphism

$\quad \infty \longmapsto N$

$\quad z \longmapsto \pi(z)$ $\overline{\mathbb{C}} \cong S^2$

ε-neighborhood of ∞ = $D_\varepsilon(\infty) = \{ z \in \overline{\mathbb{C}} \mid |z| > \frac{1}{\varepsilon} \}$



$\lim_{z \to \infty} f(z) = w_0 \iff \forall \varepsilon > 0 \ \exists R > 0$ s.t.

$\qquad |f(z) - w_0| < \varepsilon$ whenever $|z| > R$.

$\lim_{z \to z_0} f(z) = \infty \iff \forall \varepsilon > 0, \exists \delta > 0$ s.t.

$\qquad |f(z)| > \frac{1}{\varepsilon}$ whenever $0 < |z - z_0| < \delta$.

Theorem

① $\lim_{z \to z_0} f(z) = \infty \iff \lim_{z \to z_0} \frac{1}{f(z)} = 0$

② $\lim_{z \to \infty} f(z) = w_0 \iff \lim_{z \to 0} f\left(\frac{1}{z}\right) = w_0$

③ $\lim_{z \to \infty} f(z) = \infty \iff \lim_{z \to \infty} 1/f\left(\frac{1}{z}\right) = 0.$

Pf (Exercise)

Ex

1) $\lim_{z \to \infty} \frac{2z+1}{z-1}$ $\qquad \lim_{z \to \infty} \frac{2+\frac{1}{z}}{1-\frac{1}{z}} = 2$

2) $\lim_{z \to 1} \frac{2z+1}{z-1} = \infty.$

4. <u>Continuity</u> ~~Region~~ Region – domain $\cup$ {some, none, all of boundary pts of the domain}.

<u>Def</u> i) A function $f : \mathbb{C} \supset S \longrightarrow \mathbb{C}$ is continuous at $z_0 \in S$ if $\exists \lim_{z \to z_0} f(z) = f(z_0)$

ii) f is continuous in a region $R \subset \mathbb{C}$ if it is continuous at $\forall z \in R$.

<u>Fact</u>. $f, g : \mathbb{C} \supset S \longrightarrow \mathbb{C}$ are continuous at $z_0 \in S$

$\Rightarrow$ 1) $f+g,\ f \cdot g,\ \frac{f}{g}\ (g(z_0) \neq 0)$ ______ also

2) $P(z) = a_n z^n + \dots a_1 z + a_0 : \mathbb{C} \longrightarrow \mathbb{C}$ ~~are~~ is continuous in $\mathbb{C}$

<u>Theorem 1.</u> If $f : \mathbb{C} \supset A \longrightarrow B \subset \mathbb{C}$ is continuous at $z_0 \in A$

and $g : \mathbb{C} \supset B \longrightarrow \mathbb{C}$ ______ at $w_0 = f(z_0) \in B$

, then $g \circ f : A \longrightarrow \mathbb{C}$ ______ z_0.

<u>Proof</u>

Since g is cont. at w_0, $\forall \varepsilon > 0\ \exists \varepsilon_1 > 0$ s.t

$|g(w) - g(w_0)| < \varepsilon$ for all $w \in B$ with $|w - w_0| < \varepsilon_1$

Since f ______ z_0, $\exists \delta > 0$ s.t

$|f(z) - f(z_0)| < \varepsilon_1 \longleftarrow$ $z \in A$ with $|z - z_0| < \delta$.

Therefore for $\forall z \in A$ s.t. $|z - z_0| < \delta$

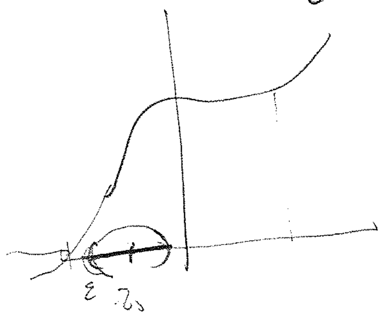
$\Rightarrow |f(z) - f(z_0)| < \varepsilon_1 \Rightarrow |g \circ f(z) - g \circ f(z_0)| < \varepsilon$

So, $g \circ f$ is cont. at $z_0 \in A$ $\boxed{\times}$

$f: \mathbb{C} \supset A \to \mathbb{C}$

Theorem 2. If $f(z)$ is continuous and $f(z_0) \neq 0$, then $f(z) \neq 0$ on some nbd of z_0 (i.e. $\exists\, \delta > 0$ s.t. $f(z) \neq 0 \ \forall z \in D_\delta(z_0)$.

Proof.



Since f is cont. at z_0 and choose

$\varepsilon_0 := \frac{|f(z_0)|}{2} > 0$, $\exists\, \delta > 0$ s.t.

$|f(z) - f(z_0)| < \varepsilon_0 \ \forall z \in D_\delta(z_0)$.

Then $|f(z)| \geq |f(z_0)| - |f(z) - f(z_0)|$

$> |f(z_0)| - \frac{|f(z_0)|}{2} = \frac{|f(z_0)|}{2} > 0.$ $\boxtimes$

Note. If $f(z) = u(x,y) + iv(x,y)$ is cont. at $z_0 = x_0 + iy_0$

then u & v are cont. at (x_0, y_0).

Proof (Exercise)

Theorem 3. If a region R is closed and bounded and (R is compact) $\to \forall \{x_n\} \subset R$, $\exists \{x_{n_k}\} \subset \{x_n\}$ s.t. $x_{n_k} \to x_0 \in R$.

if f is cont. in R, then f is bounded in R, i.e.,
$\exists M > 0$ s.t. $|f(z)| \leq M \ \forall z \in R$.

Proof f is continuous in/on $R \Rightarrow |f(z)|$ is also cont. on $R \subset \mathbb{C} = \mathbb{R}^2$

$|f(z)| = \sqrt{u^2 + v^2}$

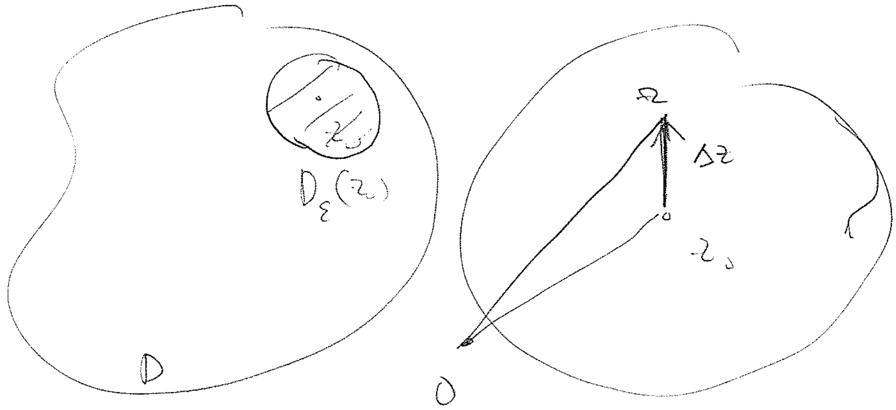
So, (known by Real Analysis) $|f(z)|$ is bounded $\boxtimes$.

$\exists \sup |f(z)| = |f(z_0)|$

& $\inf |f(z)| = |f(\tilde{z}_0)|$

5. Derivatives (도 함 수) Do ham su

Let $f : \underset{\text{a domain}}{D} \subset \mathbb{C} \longrightarrow \mathbb{C}$ and $z_0 \in D$



Define $\quad f'(z_0) := \lim_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$, $z = z_0 + \Delta z$

$$\frac{df}{dz}(z_0) = f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta f(z_0)}{\Delta z}$$

- $f'(z_0)$ is called the derivative of f at z_0.
- f is said to be differentiable at z_0.

Examples

1) $f = c \Rightarrow f' = 0$; $f = z \Rightarrow \Delta f = \Delta z$

$\dfrac{\Delta f}{\Delta z} = 1 \Rightarrow f' = 1$.

2) $f = z^2$, $z \in \mathbb{C}$

$$\frac{\Delta f(z)}{\Delta z} = \frac{f(z + \Delta z) - f(z)}{\Delta z} = \frac{(z + \Delta z)^2 - z^2}{\Delta z}$$

$$= 2z + \Delta z \longrightarrow 2z \quad \text{as } \Delta z \to 0$$

So $f'(z) = 2z$.

3) $f = z^n$, $n \geq 2$ ($\mathbb{R}^+$), $f' = n z^{n-1}$ (Exercise)

4) $f = \bar{z}$, $\frac{\Delta f}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z}$

Since $\nexists \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$, $\nexists f'(z)$.

5) $f(z) = |z|^2 = z\bar{z}$, $+ f'(0) = 0$

is cont. on $\mathbb{C}$

$+ \nexists f'(z), \forall z \neq 0$.

Fact. $\exists f'(z_0) \Longrightarrow f$ is cont. at z_0

$\not\Longleftarrow$ (counterexample $f = |z|^2$)

Proof. $|f(z) - f(z_0)| = \left|\frac{f(z) - f(z_0)}{z - z_0}\right| \cdot |z - z_0| \to 0$ as $z \to z_0$

$\left|\frac{f(z) - f(z_0)}{z - z_0}\right| \to f'(z_0)$, $|z - z_0| \to 0$

$\boxtimes$

6. Differentiation formulas

1) $\frac{d}{dz} c = 0$, $\frac{d}{dz} z = 1$, $\frac{d}{dz} z^n = n z^{n-1}$ $(n \geq 1)$

2) $\frac{d}{dz}[c \cdot f(z)] = c \cdot f'(z)$

3) $\frac{d}{dz}[f(z) \pm g(z)] = f'(z) \pm g'(z)$

4) $\frac{d}{dz}[f(z) \cdot g(z)] = f(z) \cdot g'(z) + f'(z) g(z)$

5) $\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{g(z) \cdot f'(z) - f(z) \cdot g'(z)}{(g(z))^2}$ if $g(z) \neq 0$

Chain rule

$f : \mathbb{C} \supset A \longrightarrow B \subset \mathbb{C}$, $\exists f'(z_0)$, $z_0 \in A$

$g : \mathbb{C} \supset B \longrightarrow \mathbb{C}$, $\exists g'(w_0)$, $w_0 = f(z_0)$

Then $(g \circ f)'(z_0) = g'(f(z_0)) \cdot f'(z_0)$

Proof (Exercise)

7. Cauchy – Riemann equations (Augustin Louis Cauchy 1789–1857 (67 years old), Bernhard Riemann 1826 – 1866 (39 years old))

$w = f : \mathbb{C} \supset D \longrightarrow \mathbb{C}$, $f = u + iv$

(D: a domain)

Theorem 1. Suppose that $f(z) = u(x,y) + i v(x,y)$ and $\exists\, f'(z_0)$ at $z_0 = x_0 + i y_0$. Then

$\exists\ u_x(z_0), u_y(z_0), v_x(z_0), v_y(z_0)$, and they satisfy the Cauchy-Riemann equation

$$\begin{cases} u_x(x_0,y_0) = v_y(x_0,y_0) \\ u_y(x_0,y_0) = -v_x(x_0,y_0) \end{cases}$$

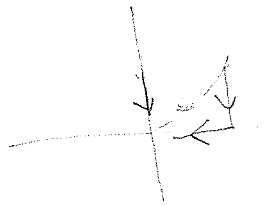
Also $f'(z_0) = u_x(x_0,y_0) + i v_x(x_0,y_0)$

Proof

$$\Delta z = \Delta x + i \Delta y$$

$$\begin{aligned} \Delta w &= f(z_0 + \Delta z) - f(z_0) \\ &= u(x_0+\Delta x, y_0+\Delta y) - u(x_0, y_0) \\ &\quad + i\left[v(x_0+\Delta x, y_0+\Delta y) - v(x_0,y_0)\right] \end{aligned}$$

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{(\Delta x, \Delta y)\to(0,0)} \frac{\Delta w}{\Delta z}$$



$$\frac{\Delta w}{\Delta z} = \frac{u(x_0+\Delta x, y_0+\Delta y) - u(x_0,y_0)}{}$$

$$\Rightarrow \begin{cases} f'(z_0) = \lim\limits_{\Delta z = (\Delta x, 0)\to(0,0)} \dfrac{\Delta w}{\Delta z} = \lim\limits_{\Delta x \to 0} \dfrac{\Delta w}{\Delta x} & (1) \\ f'(z_0) = \lim\limits_{\Delta z = (0,\Delta y)\to(0,0)} \dfrac{\Delta w}{\Delta z} = \lim\limits_{\Delta y \to 0} \dfrac{\Delta w}{i \Delta y} & (2) \end{cases}$$

$$(1):\ \frac{\Delta w}{\Delta x} = \frac{u(x_0+\Delta x, y_0) - u(x_0,y_0)}{\Delta x} + i\, \frac{v(x_0+\Delta x, y_0) - v(x_0,y_0)}{\Delta x}$$

$$\downarrow \qquad\qquad \downarrow$$

$$f'(z_0) = \exists\ u'_x(x_0,y_0) + i\, v'_x \qquad (3)$$

② $\frac{\Delta w}{i\Delta y} = \frac{u(x_0, y_0+\Delta y) - u(x_0,y_0)}{i\Delta y} + i\,\frac{v(x_0, y_0+\Delta y) - v(x_0,y_0)}{i\Delta y}$

$\Delta y \to 0$

$f'(z_0) = \frac{1}{i}\,\exists u'_y + \exists v'_y(x_0,y_0)$ (4)

By (3) & (4), we have
$$\begin{cases} u'_x(x_0,y_0) = v'_y(x_0,y_0) \\ u'_y(x_0,y_0) = -v'_x(x_0,y_0) \end{cases}$$

Examples

1) $f(z) = z^2 = x^2 - y^2 + i2xy$

$u = x^2 - y^2$; $u'_x = 2x = v'_y$

$v = 2xy$; $u'_y = -2y = -v'_x$

f satisfies the C-R eq.

2) $f(z) = \bar{z} = x - iy$, $u(x,y) = x$, $v(x,y) = -y$

$u'_x = 1 \neq -1 = v'_y$

f does not satisfy the C-R eq. $\rightarrow \nexists f'(z)$

3) $f(z) = |z|^2$, $u = x^2+y^2$, $v = 0$; $u'_x = 2x$, $u'_y = 2y$

C-R eq. holds $\Longleftrightarrow \begin{cases} u'_x = 2x = 0 \\ u'_y = 2y = 0 \end{cases} \Longleftrightarrow (x,y) = (0,0)$.

$\Rightarrow \nexists f'(z)$ for $z \neq 0$.

$\exists f'(z_0) \Longrightarrow$ C-R eq.

Note: C-R eq. $\not\Longrightarrow \exists f'(z_0)$?

$\underbrace{\frac{x^3-3xy^2}{x^2+y^2}}_{u} + i\,\underbrace{\frac{y^3-3x^2y}{x^2+y^2}}_{v}$

Counter-example: $f(z) = \begin{cases} \frac{\bar{z}^2}{z} & \text{when } z \neq 0 \\ 0 & z = 0 \end{cases}$

$\begin{cases} u_x(0) = v_y(0) = 1 \\ u_y(0) = -v_x(0) = 0 \end{cases}$

C-R eq. holds at 0 but $\nexists f'(0)$.

7'. Sufficient conditions for differentiability

__Theorem 2.__ Suppose that $f: D_\varepsilon(z_0) \longrightarrow \mathbb{C}$, $f(z) = u(x,y) + i v(x,y)$

and $\exists\ u'_x, u'_y, v'_x, v'_y$ on $D_\varepsilon(z_0)$. If u'_x, u'_y, v'_x, v'_y are continuous at (x_0, y_0) & satisfy the C-R equations

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \text{at } (x_0, y_0),$$

then $\exists\ f'(z_0)$.

__Proof.__ $\Delta z = \Delta x + i\Delta y$, $\Delta w = \Delta u + i \Delta v$, $|\Delta z| = \sqrt{(\Delta x)^2 + (\Delta y)^2}$

$\Delta u = u(x+\Delta x, y + \Delta y) - u(x,y)$

$\Delta v = v(\ldots) - v(\ldots)$

Since $\exists\ u'_x, u'_y, v'_x, v'_y$ are conti. at z_0, we can write (by Calculus)

$(u \in C^1, v \in C^1)$

$$\Delta u = u_x(x_0,y_0)\Delta x + u_y(x_0,y_0)\cdot \Delta y + \varepsilon_1 \cdot \left(\sqrt{\Delta x^2 + \Delta y^2}\right), \text{ (Taylor's theorem)}$$

$$\Delta v = v_x(x_0,y_0)\Delta x + v_y(x_0,y_0)\Delta y + \varepsilon_2 \cdot \left(\sqrt{\Delta x^2 + \Delta y^2}\right)$$

, where $\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$ as $\Delta z \to 0$

$$\Delta w = \Delta u + i \Delta v$$

$$\Rightarrow \Delta w = u_x(x_0,y_0)\Delta x + \overset{= -v_x}{u_y}(x_0,y_0)\Delta y + \varepsilon_1(|\Delta z|) +$$

$$+ i\left[v_x(x_0,y_0)\Delta x + \underset{\substack{\parallel \\ u_x}}{v_y}(x_0,y_0)\Delta y + \varepsilon_2(|\Delta z|)\right]$$

$$= u_x(x_0,y_0)(\Delta x + i \Delta y) + i v_x(i\Delta x - \Delta y) + (\varepsilon_1 + i\varepsilon_2)\cdot|\Delta z|$$

$$\frac{\Delta w}{\Delta z} = u_x(x_0,y_0) + i v_x(x_0,y_0) + \underbrace{(\varepsilon_1 + i\varepsilon_2)}_{\to 0} \underbrace{\frac{|\Delta z|}{\Delta z}}_{\text{bounded}}$$

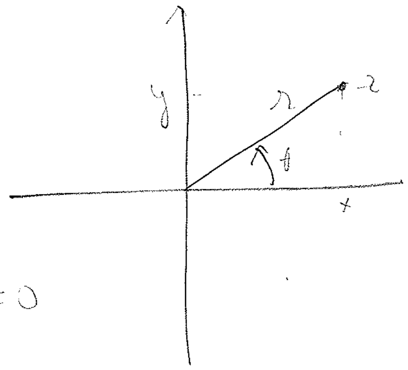
$$\Rightarrow \exists \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = u_x(x_0,y_0) + i v_x(x_0,y_0) \quad \square$$

__Example__ $f(z) = |z|^2$, $\exists\ u'_x(0,0) = v'_y(0,0) = 0$

$u'_y(0,0) = -v'_x(0,0) = 0$

& $u, v \in C^1 \xrightarrow{\text{by Thm 2}} \exists\ f'(0)$.

8. Polar Coordinates



$z = x + iy = re^{i\theta} \neq 0$

$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

By the chain rule,

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} = u'_x\cos\theta + \underbrace{u'_y}_{=-v'_x}\sin\theta$$

$$\frac{\partial u}{\partial \theta} = u'_x\cdot\frac{\partial x}{\partial \theta} + u'_y\cdot\frac{\partial y}{\partial \theta} = u'_x(-r\sin\theta) + u'_y(r\cos\theta)$$

$$-r \times \Big| \frac{\partial v}{\partial r} = \qquad = v'_x\cos\theta + \underbrace{v'_y}_{=-v_x}\sin\theta$$

$$\frac{\partial v}{\partial \theta} = \qquad = v'_x(-r\sin\theta) + v'_y(r\cos\theta)$$

We have:

$$\begin{cases} u'_x = v_y \\ u'_y = -v_x \end{cases} \overset{\text{check}}{\Longleftrightarrow} \begin{cases} \dfrac{\partial u}{\partial \theta} = -r\dfrac{\partial v}{\partial r} \\ r\dfrac{\partial u}{\partial r} = \dfrac{\partial v}{\partial \theta} \end{cases} \qquad \begin{cases} r u_r = v_\theta \\ u_\theta = -r v_r \end{cases}$$

$z \in \mathbb{C} \mid |z - z_0| < \varepsilon\}$

Theorem 2 $\Longleftrightarrow$

Theorem 2'. $f(z) = u(r,\theta) + i v(r,\theta) : D_\varepsilon(z_0) \longrightarrow \mathbb{C}$, $\varepsilon_0 > 0$

Suppose that
1) $\exists\ u'_r, u'_\theta, v'_r, v'_\theta$ on $D_\varepsilon(z_0)$
2) ______ are continuous and satisfy the C–R eq. $\begin{cases} r u_r = v_\theta \\ u_\theta = -r v_r \end{cases}$ at z_0.

Then $\exists\ f'(z_0) = e^{-i\theta}(u_r(z_0) + i v_r(z_0))$

Ex 1) $f = \frac{1}{z} = \frac{1}{r e^{i\theta}} = \frac{e^{-i\theta}}{r} = \frac{\cos\theta}{r} - i\frac{\sin\theta}{r}$

$u = \frac{\cos\theta}{r}, \quad v = \frac{\sin\theta}{r}$

$$\begin{cases} r u_r = -\frac{\cos\theta}{r} = v_\theta \\ u_\theta = -\frac{\sin\theta}{r} = -r \cdot v_r \end{cases}$$

$\Rightarrow \exists f'(z) = e^{-i\theta}(u_r + i v_r) = -\frac{e^{-i\theta} \cdot e^{-i\theta}}{r^2} = -\frac{1}{z^2}$

$u_r = -\frac{\cos\theta}{r^2}, \quad v_r = \frac{\sin\theta}{r^2}$, $\forall z \neq 0$.

2) $f = \sqrt[3]{r}\, e^{i\theta/3} = z^{1/3}$ $\quad \alpha < \theta < \alpha + 2\pi$
(Exercise) $\quad r > 0$.

$$f' = \frac{1}{3(z^{1/3})^2}$$

3) $f = \operatorname{Log} z = \ln r + i\theta$ $\quad , 0 < \theta < 2\pi$

$u = \ln r \qquad u'_r = \frac{1}{r}, \quad u_\theta = 0$
$v = \theta \qquad v_r = 0, \quad v_\theta = 1$

$$\begin{cases} r u_r = v_\theta \\ u_\theta = -r v_r \end{cases}$$

$\Rightarrow \exists f' = e^{-i\theta}(u_r + i v_r)$
$= e^{-i\theta}\left(\frac{1}{r} + 0\right) = \frac{1}{r e^{i\theta}} = \frac{1}{z}$.

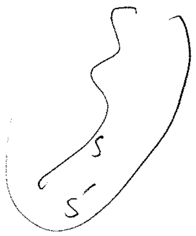
9. Analytic functions

Def. i) $f : \mathbb{C} \supset D \longrightarrow \mathbb{C}$ is analytic at $z_0 \in D$
(o domain) (holomorphic)
if it has a derivative at $\forall z$ in some nbd of z_0
($\exists f'(z)$ $\forall z \in D_\varepsilon(z_0)$, $\varepsilon > 0$).

ii) f is analytic in open set D if it is analytic at $\forall z \in D$

iii) f is analytic in a set S if it is analytic in an open set $D \supset S$.

Ex.



1) $f = \frac{1}{z}$ is analytic in $\mathbb{C} \setminus \{0\}$

2) $f = |z|^2$ is not analytic at any pt.

3) $f = P(z)$ is analytic in $\mathbb{C}$

3) $f = \frac{P(z)}{Q(z)}$ ... in $\mathbb{C} \setminus Z(Q)$

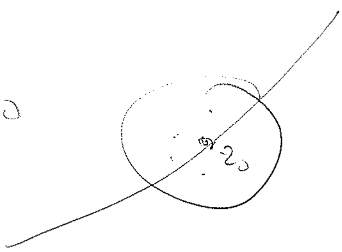
$Z(Q) = \{ z \in \mathbb{C} \mid Q(z) = 0 \}$

Def. An entire function is a ftion that is analytic in $\mathbb{C}$.

Ex. $P(z)$ is an entire ftion.

Def. z_0 is a singular point of f if f fails to
(singularity)
be analytic at z_0 but is analytic at some pt in $\forall$ nbd of z_0.

Ex



1) $f = \frac{1}{z}$ has a singular pt $z = 0$

2) $f = |z|^2$ has no singular pt

3) $f = e^{\frac{1}{z}}$, ...

<u>Fact</u>

1) f is analytic in a domain D $\Rightarrow$ f is cont. in D

2) f — — $\Rightarrow$ C-R eq holds in D

3) f, g are — — — $\Leftarrow$, , $g \in C^1$

$\Rightarrow$ $f \pm g$, $f \cdot g$, $\frac{f}{g}$ are analytic in D.

($g \neq 0$ in D)

4) $f : \mathbb{C} \supset D \to \Omega \subset \mathbb{C}$ are analytic in D
$g : \Omega \to \mathbb{C}$ in Ω

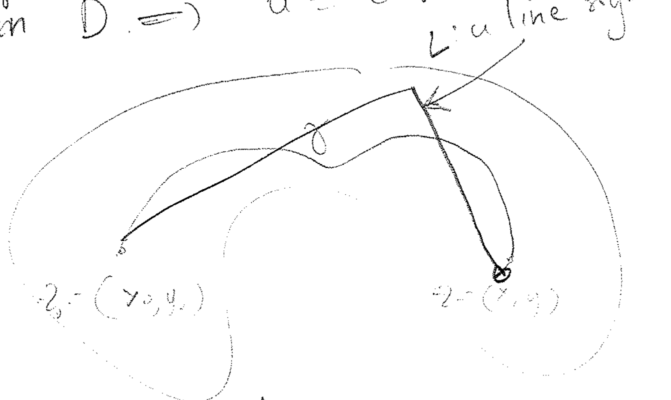
$\Rightarrow$ $g \circ f$ are analytic in D.

<u>Theorem</u>. If $f'(z) = 0$ $\forall z \in \underset{\text{a domain}}{D}$, then $f = $ const.

Proof. $f'(z) = 0 \Rightarrow u_x = v_x = u_y = v_y = 0$ (by C-R eqt.)

$$\Rightarrow \begin{array}{l} u = \text{const} \\ v = \text{—} \end{array} \Rightarrow f = \text{const} \ \square$$

<u>Claim</u> $u_x = u_y = 0$ in $D \Rightarrow u = $ const



Proof (Exercise - in text book)

<u>Fix</u> $z_0 = (x_0, y_0) \in D$

Let $z = (x, y) \in D$ be any pt in D.

Let $\gamma : [0, 1] \to \mathbb{C}$ is a C^1-smooth curve s.t
$(-\varepsilon, 1+\varepsilon)$
$\gamma(t) = x(t) + i y(t)$ $\qquad \gamma(0) = z_0$, $\gamma(1) = z$.

Then let $g(t) := u \circ \gamma(t)$. Then
$= u(x(t), y(t))$

$$g'(t) = u'_x \cdot x'(t) + u'_y \cdot y'(t) \equiv 0 \text{ on } [0, 1]$$

$\Rightarrow g = $ const $\Rightarrow g(1) = g(0)$
$\Rightarrow u(x, y) = u(x_0, y_0)$

$g(t) - g(t') = g'(t^*)(t - t') \equiv 0$
$\Rightarrow g(t) = g(t_0) = $ const $\forall t$
$\boxed{\times}$

Thus, $u = $ const.

Ex. 1. $f(z)$ and $\overline{f(z)}$ are both analytic in a domain $\mathbb{D}$

$\Rightarrow$ $f = \text{const}$ on $\mathbb{D}$

Proof $f = u + iv$, $\bar{f} = u - iv$

Since $f, \bar{f}$ are analytic in $\mathbb{D}$, C-R eq. holds

i.e., $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$ & $\begin{cases} u_x = -v_y \\ u_y = +v_x \end{cases}$

$\Rightarrow$ $v_x = -v_x \Rightarrow v_x = 0 \Rightarrow u_y = 0$

$v_y = -v_y \Rightarrow v_y = 0 \Rightarrow u_x = 0$

$\Rightarrow$ $u = \text{const}$, $v = \text{const}$ $\Rightarrow$ $f = \text{const}$.

Ex. 2. f is analytic in $\mathbb{D}$ & $|f(z)| = \text{const}$ in $\mathbb{D}$ $\Rightarrow$

$\Rightarrow$ $f = \text{const}$.

Proof. $|f(z)| = c$ $\forall z \in \mathbb{D}$

+ $c = 0 \Rightarrow f = 0$.

+ $c > 0 \Rightarrow$ $|f(z)|^2 = c^2 \Rightarrow f(z)\overline{f(z)} = c^2$

$\Rightarrow \begin{cases} f(z) \neq 0 \quad \forall z \in \mathbb{D} \\ \overline{f(z)} = \dfrac{c^2}{f(z)} \text{ is analytic in } \mathbb{D} \end{cases}$

Thus, f & $\bar{f}$ are both analytic in $\mathbb{D}$ $\overset{\text{Ex.1}}{\Rightarrow}$ $f = \text{const}$.

10. Harmonic functions

Def. $H : \mathbb{C} \supset D \longrightarrow \mathbb{R}$ is harmonic in D if
(D a domain)

$H \in C^2(D)$ (i.e., it has continuous partial derivatives of the 2nd order)

and satisfy $\Delta H = H_{xx} + H_{yy} = 0$, $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ (the Laplace operator)

Ex : $H(x,y) = x^2 - y^2$; xy

Theorem 1. If $f = u + iv : D \to \mathbb{C}$ is analytic, then u & v are harmonic in D

Proof $f : D \to \mathbb{C}$ is analytic

- Fact : $u, v \in C^\infty(D) = \{ u : D \to \mathbb{C} \mid \exists \frac{\partial^{m+n} u}{\partial x^m \partial y^n} \ \forall m, n \in \mathbb{N} \}$
(we will prove this fact later)

- Since f is analytic, C-R eq. holds

$$\begin{cases} u'_x = v_y \\ u_y = -v_x \end{cases} \Rightarrow \begin{cases} u''_{xx} = v_{yx} \\ u''_{yy} = -v_{xy} \end{cases} \Rightarrow \Delta u = 0$$

Similarly, $\Delta v = 0$ $\boxed{\times}$

Examples

1) $f(z) = \frac{i}{z^2} = u + iv \Rightarrow u(x,y) = \frac{2xy}{(x^2+y^2)^2}, v = \frac{x^2-y^2}{(x^2+y^2)^2}$

2) $f(z) = \frac{1}{z} = u + iv = \frac{\bar{z}}{|z|^2} = \frac{x - iy}{(x^2+y^2)^2}$

$\Rightarrow u(x,y) = \frac{x}{(x^2+y^2)^2}$, $v(x,y) = \frac{-y}{(x^2+y^2)^2}$

3) $f(z) = ze^z$ . . . . .

Def. u, v are harmonic in a domain $D \subset \mathbb{C}$.
v is said to be a harmonic conjugate of u if their first-order partial derivatives satisfy the Cauchy-Riemann equation on D

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \text{ on } D.$$

Theorem 2. A function $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain $D \subset \mathbb{C}$ if and only if v is a harmonic conjugate of u.

Proof (easy!)

Note. 1/ $f \in Hol(D) \Rightarrow f \in C^\infty(D)$ & $Re f \in H(D)$

2/ If $u \in H(D) \Rightarrow \exists f \in Hol(D)$ s.t. $u = Re f$.
& $u \in C^\infty(D)$ (chapter 9).

Example

$u := x^2 - y^2$, $\Delta u = 0 \Rightarrow u$ is harmonic in $\mathbb{C}$

Find a harmonic conjugate of u?

—

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \Rightarrow \begin{cases} v_x = -u_y = +2y \\ v_y = u_x = 2x \end{cases}$$

$$\Rightarrow \quad v(x,y) = 2xy + \varphi(y)$$

$$v_y = 2x + \varphi'(y) = 2x$$

$$\Rightarrow \quad \varphi'(y) = 0 \Rightarrow \varphi(y) \equiv c = \text{const}.$$

Thus $v(x,y) = 2xy$.

$$\& \cdot f(z) = u + iv = x^2 - y^2 + 2ixy + ic$$
$$= z^2 + ic.$$

$\cdot\ u = Re f$.

—

11. Uniquely determined analytic functions

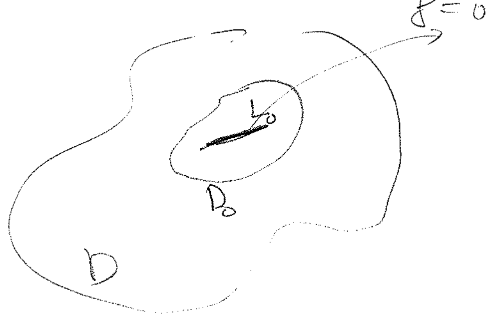
$D \subset \mathbb{C}$ is a domain.

(Identity theorem) **Lemma** Suppose that

a) $f \in Hol(D)$ (f is analytic in D)

b) $f(z) = 0$ at $\forall z \in$ a subdomain or a line segment contained in D.

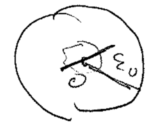
Then $f(z) \equiv 0$ in D.



We need the following fact.

Fact 1 (Thm 3, Sec. 75, p. 251)

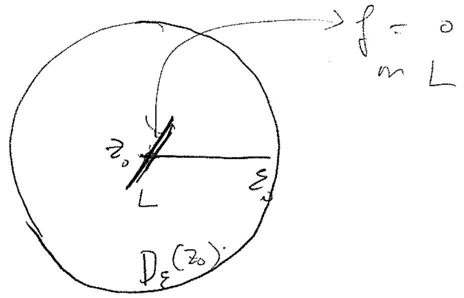
If $f \in Hol(D_\varepsilon(z_0))$ and ($\varepsilon > 0$)

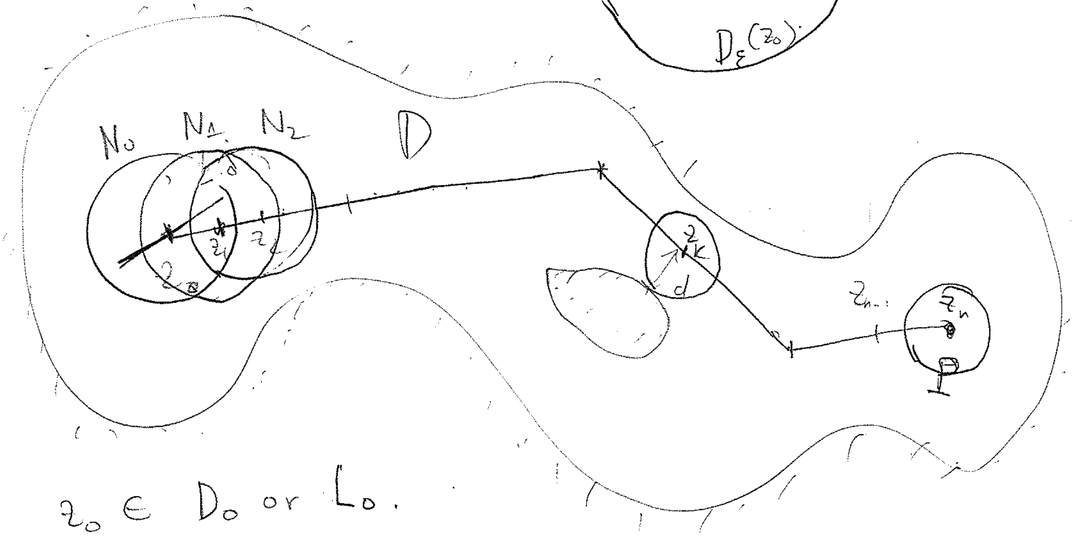


if $f(z) = 0$ at each point z of a subdomain or a line segment L containing z_0, then $f(z) \equiv 0$ in $D_\varepsilon(z_0)$.

Proof of Lemma

Suppose that $f \equiv 0$ on D_0 or L_0





Fix $z_0 \in D_0$ or L_0.

Let $P \in D$ be an arbitrary point.

We will show that $f(P) = 0$. Thus $f(z) \equiv 0$ in D.

Since D is connected, $\exists$ a polygonal line L, consisting of a finite number of line segments joined end to end and $L \subset D$.

Denote $d = \operatorname{dist}(L, \partial D)$ (if $D = \mathbb{C}$, then $d := 1$).

Then $\exists\ z_0, z_1, \dots, z_n = P$ $(z_j \in L)$
such that $|z_k - z_{k-1}| < d$ $(k = 1, 2, \dots, n)$.

$N_k := D_d(z_k) = \{z \in \mathbb{C} : |z - z_k| < d\} \subset D$

<u>Note that</u> $z_{k+1} \in D_d(z_k) = N_k \ \forall\ k = 0, 1, 2, \dots, n-1$.

Since $f \in Hol(N_0)$ & $f = 0$ in $\underset{z_0}{\underset{\cup}{D_0}}$ or $\underset{z_0}{\underset{\cup}{L_0}}$,
by Fact 1 we have $f \equiv 0$ in N_0.
But $z_1 \in N_0$, by Fact 1 we get $f \equiv 0$ in N_1.

. . . .

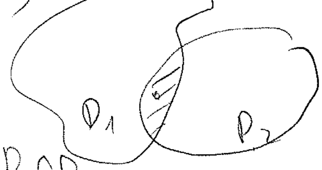
Finally, $f \equiv 0$ in $\underset{P}{\underset{\cup}{N_{n-1}}}$ and thus $f(P) = 0$ $\boxtimes$

<u>Remark</u>. Suppose that $f_1, f_2 \in Hol(D)$ and
$f_1 \equiv f_2$ in D_0 or L_0 ($D_0 \underset{\text{subdomain}}{\subset} D$ & L_0 : a line segment $\subset D$).

Let $f(z) := f_1(z) - f_2(z)$. Then $f \in Hol(D)$
& $f \equiv 0$ in D_0 or L_0. By Lemma, $f \equiv 0$ in D.
Thus $f_1(z) \equiv f_2$ in D.

<u>Theorem</u>. A function that is analytic in $D \subset \mathbb{C}$ is uniquely determined over D by its values in a (sub) domain or along a line segment, contained in D.

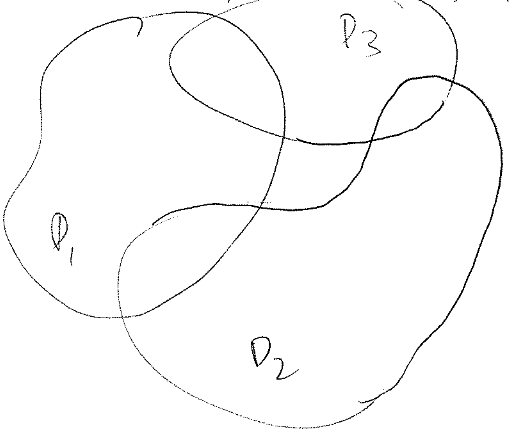
<u>Def</u>. $f_1 \in Hol(D_1)$, $f_2 \in Hol(D_2)$, $D_1 \cap D_2 \neq \phi$



f_2 is an analytic continuation of f_1 from D_1 into D_2 if $f_2(z) = f_1(z)\ \forall z \in D_1 \cap D_2$.

<u>Remark</u>. By Thm, whenever that analytic continuation exists, it is unique.

$D_1 \cap D_2 \neq \phi$, $D_2 \cap D_3 \neq \phi$, $D_1 \cap D_3 \neq \phi$



Note. $f_j \in Hol(D_j)$ $(j = 1,2,3)$

- f_2 is an analytic continuation of f_1 from D_1 into D_2.
- f_3 ———————— D_3.

———————

- But it is not necessarily true that $f_3(z) = f_2(z)$ $\forall z \in D_2 \cap D_3$

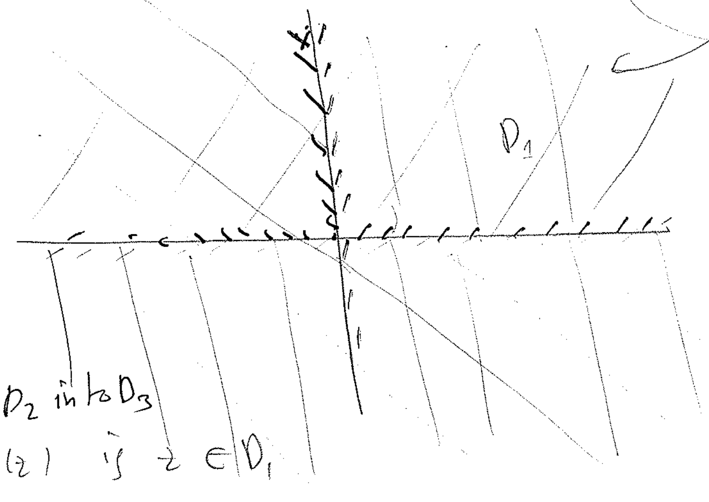
Counterexample (Exercise 2, p. 87).

$f_1(z) = \sqrt{r}\, e^{i\theta/2}$ $(r > 0,\ 0 < \theta < \pi)$

$f_2(z) = \sqrt{r}\, e^{i\theta/2}$ $(r > 0,\ \frac{\pi}{2} < \theta < 2\pi)$

$f_3(z) = \sqrt{r}\, e^{i\theta/2}$ $(r > 0,\ \pi < \theta < \frac{5\pi}{2})$

$f_3(z) = -f_1(z)$ in $0 < \theta < \frac{\pi}{2}$



- if f_2 is an analytic continuation of f_1 from D_1 into D_2
- f_3 is an analytic continuation of f_2 from D_2 into D_3

if, then

$$F(z) := \begin{cases} f_1(z) & \text{if } z \in D_1 \\ f_2(z) & \text{if } z \in D_2 \end{cases}$$

Then $F \in Hol(D_1 \cup D_2)$.

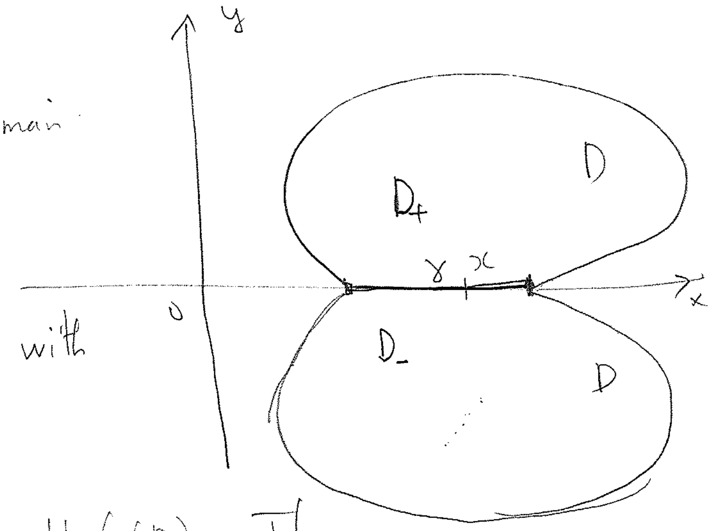
F is the analytic continuation into $D_1 \cup D_2$ of either f_1 or f_2 (f_1, f_2 are called elements of F).

12. **Reflection Principle**

$D = D_+ \cup D_- \cup \gamma$: a domain

γ : a segment of x-axis

D_- is the reflection of D_+ with respect to x-axis.



Theorem. Suppose $f \in Hol(D)$. Then

$$\overline{f(z)} = f(\bar z) \ \forall z \in D \iff f(\gamma) \subset \mathbb{R}.$$

Proof Note $\overline{f(z)} = f(\bar z) \iff f(z) = \overline{f(\bar z)}$

"$\Leftarrow$" Suppose that $f(\gamma) \subset \mathbb{R}$, i.e., $f(x) \in \mathbb{R} \ \forall x \in \gamma$.

We will prove that $f(z) = \overline{f(\bar z)}$

Let $F(z) := \overline{f(\bar z)} \ \forall z \in D$

⊕ $F \in Hol(D)$

$f(z) = u(x,y) + iv(x,y)$, $\overline{f(\bar z)} = u(x,-y) - iv(x,-y)$.

$F(z) = U(x,y) + iV(x,y)$

$$F(z) = \overline{f(\bar z)} \iff \begin{cases} U(x,y) = u(x,-y) = u(x,t) \\ V(x,y) = -v(x,-y) = -v(x,t) \end{cases}$$

with $t = -y$

$U_x(x,y) = u_x(x,t)$; $U_y = -u_t(x,t)$

$V_x(x,y) = -v_x(x,t)$; $V_y = v_t(x,t)$

$$\Rightarrow \begin{cases} U_x(x,y) = V_y(x,y) \\ U_y(x,y) = -V_x(x,y) \end{cases} \ \forall (x,y) \in D$$

$\Rightarrow F \in C^1(D)$ and satisfies the Cauchy-Riemann equation

Thus $F \in Hol(D)$ (Theorem 2 in Sec.?!)

Chapter 3. Elementary functions

29. The exponential function

$$f(z) = e^z = e^x \cdot e^{iy} = e^x(\cos y + i\sin y) \quad \forall z \in \mathbb{C}$$

$$f: \mathbb{C} \to \mathbb{C} \;.\; N$$

<u>Properties</u>

1) $e^{z_1} e^{z_2} = e^{z_1+z_2}$; $\dfrac{e^{z_1}}{e^{z_2}} = e^{z_1 - z_2}$

2) $\dfrac{d}{dz} e^z = e^z$

3) $e^z \neq 0 \;\forall z \in \mathbb{C}$ $\left(|e^z| = e^x > 0 \;\forall x \in \mathbb{R} \right)$

4) $e^{z+2k\pi i} = e^z$ (periodic)

5) $e^{\pi i} = -1$.

<u>Example</u>. $e^i = \cos 1 + i\sin 1$

Find z s.t. : $e^{z} = i \iff e^x \cdot e^{iy} = i = e^{i\pi/4}$

$$\Rightarrow \begin{cases} x = 0 \\ y = \frac{\pi}{4} + 2k\pi \quad \forall k \in \mathbb{Z} \end{cases}$$

So $z = i\left(\frac{\pi}{4} + 2k\pi\right)$.

30. The logarithmic function

$e^w = z$, $z = re^{i\theta_0}$, $r > 0$. $\theta_0 = \operatorname{Arg} z \in (-\pi, \pi]$

$w = x + iy$.

$$e^w = z \iff e^x \cdot e^{iy} = re^{i\theta_0} \iff \begin{cases} x = \log r = \ln r \\ y = \theta_0 + 2n\pi, \; n \in \mathbb{Z} \end{cases}$$

$$\log(z) = \log r + i(\theta_0 + 2n\pi) = \log|z| + i(\operatorname{Arg} z + 2n\pi)$$
$$= \operatorname{Ln} z + i2n\pi$$

$\operatorname{Ln} z := \log r + i\theta_0 = \ln|z| + i\operatorname{Arg} z$:

the principle branch of $\log z$.

Ex. $\log i = \ln|i| + i\left(\frac{\pi}{2} + 2n\pi\right)$
$= i\left(\frac{\pi}{2} + 2n\pi\right), n \in \mathbb{Z}$.

$\log(1+i) = \ln\sqrt{2} + i\left(\frac{\pi}{4} + 2n\pi\right), n \in \mathbb{Z}$

$\log 1 = 2n i\pi$

$\log(-1) = (2n+1)\pi i$.

31. Branches and derivatives of logarithms

$z = re^{i\theta_0}, \quad \theta_0 = \mathrm{Arg}\, z$

$\log z = \ln r + i(\theta_0 + 2n\pi), n = 0, \pm 1, \pm 2, \cdots$ ← multi-valued function

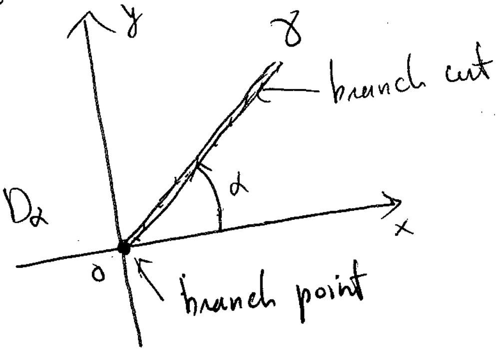
$= \ln r + i\theta, \quad \theta = \arg z$

$D_\alpha := \{ z \in \mathbb{C}^* \mid r > 0, \ \alpha < \theta < \alpha + 2\pi \}, \quad \alpha \in \mathbb{R}$.

~~$f(z) := \log z =$~~

$f_\alpha : D_\alpha \longrightarrow \mathbb{C}$ ← single-valued function & $\in C^0(D_\alpha)$

$z \longmapsto f_\alpha(z) = \ln r + i\theta, \quad r > 0, \ \alpha < \theta < \alpha + 2\pi$
$= \ln z$



$f_\alpha = u + iv$

$u(r,\theta) = \ln r$

$v(r,\theta) = \theta$

$u_r = \frac{1}{r}, \quad u_\theta = 0$

$v_r = 0, \quad v_\theta = 1$

$\hookrightarrow \begin{cases} r u_r = v_\theta \\ u_\theta = -r v_r \end{cases}$

$\Rightarrow f_\alpha$ satisfies the Cauchy-Riemann equations.

Thus $f_\alpha \in \mathrm{Hol}(D_\alpha)$ and $\frac{d}{dz} f_\alpha(z) = e^{-i\theta}(u_r + i v_r) = \frac{1}{re^{i\theta}} = \frac{1}{z}$

- $\dfrac{d}{dz} \log z = \dfrac{1}{z} \quad \forall\, z \in D_\alpha \qquad (\log z = \operatorname{Log} z + 2n\pi i,\ n \in \mathbb{Z})$

- $\dfrac{d}{dz} \operatorname{Log} z = \dfrac{1}{z} \quad \forall\, z \in D_{-\pi} = \{ z \mid r > 0\ \ -\pi < \theta < \pi \}$

<u>Def.</u> A branch of a multi-valued function f is any single-valued function F that is analytic in some domain at each point z of which the value $F(z)$ is one of the values of $f(z)$.

Example

1) f_α is a branch of $\log z$ $(\alpha \in \mathbb{R})$

2) $f_{-\pi}(z) = \operatorname{Log} z$ is called the principle branch of $\log z$.

<u>Def.</u> i) A branch cut γ is a portion of a line or a curve that is introduced in order to define a branch F of a multiple-valued function f.

ii) $p \in \gamma$ is a singular point

iii) A branch point is $p \in \gamma$ for all branch cut γ.

<u>Example</u> 1) 0 is a branch point of $\log z$.

2) $\operatorname{Log}(i^3) = \operatorname{Log}(-i) = -\dfrac{\pi}{2} i \quad \Big| \rightarrow \operatorname{Log}(i^3) \neq 3 \operatorname{Log} i$

$3 \operatorname{Log} i = 3 \dfrac{\pi}{2} i$

32. Some identities involving logarithms

a) $$\log(z_1 z_2) = \log z_1 + \log z_2 \quad \forall\, z_1, z_2 \in \mathbb{C}^{*} = \{z \mid z \neq 0\}.$$

Proof.

Note $\arg(z_1 z_2) = \arg z_1 + \arg z_2$

$$\log(z_1 z_2) = \ln|z_1 z_2| + i\arg(z_1 z_2)$$
$$= \ln|z_1| + \ln|z_2| + i(\arg z_1 + \arg z_2)$$
$$= \log z_1 + \log z_2 .$$

<u>Example</u>. $\operatorname{Log}(z_1 z_2) \neq \operatorname{Log} z_1 + \operatorname{Log} z_2$

<u>Counterexample</u>

$$\operatorname{Log}((-1)\cdot(-1)) = \operatorname{Log}(+1) = 0$$
$$\operatorname{Log}(-1) = \pi i$$
$$0 = \operatorname{Log}((-1)(-1)) \neq 2\pi i = \operatorname{Log}(-1) + \operatorname{Log}(-1).$$

b) $$\log\frac{z_1}{z_2} = \cancel{\frac{\log z_1}{\log z_2}} \quad (\text{Exercise})$$

$$\log\frac{z_1}{z_2} = \log z_1 - \log z_2 \quad (\text{Exercise})$$

c) $$z^n = e^{n\log z}, \quad n = 0, \pm 1, \pm 2, \dots, \quad \forall\, z \in \mathbb{C}^{*}.$$

Proof $z = re^{i\theta_0}$, $\quad z^n = r^n e^{in\theta_0}$ (1), $\quad \theta_0 = \operatorname{Arg} z$.

$$\log z = \ln r + i(\theta_0 + 2k\pi),\ k \in \mathbb{Z}$$

$$e^{n\log z} = e^{n\log r}\cdot e^{n i\theta_0}\cdot \underbrace{e^{n i 2k\pi}}_{=1} = r^n \cdot e^{in\theta_0} \quad (2)$$

(1) & (2) we get $z^n = e^{n\log z}$.

$\sqrt[n]{z} = e^{\frac{1}{n}\log z}$.

$z = re^{i\theta_0}$, $\theta_0 = \operatorname{Arg} z$.

$\log z = \ln r + i(\theta_0 + 2k\pi)$, $k \in \mathbb{Z}$

$$e^{\frac{1}{n}\log z} = e^{\frac{\log r}{n} + i\left(\frac{\theta_0 + 2k\pi}{n}\right)} = \sqrt[n]{r}\, e^{i\frac{\theta_0 + 2k\pi}{n}} = \sqrt[n]{z}.$$

$(k = 0, 1, \dots, n-1)$.

33. Complex exponents.

$z \neq 0$, $c \in \mathbb{C}$

$z^c := e^{c\log z}$.

Ex. $1^i = e^{i\log 1} = e^{i(0 + 2n\pi i)} = e^{-2n\pi}$, $n \in \mathbb{Z}$

Properties:

1) $\frac{1}{z^c} = z^{-c}$ $\quad$ ($e^{-z} = 1/e^z$)

2) $\frac{d}{dz} z^c = c z^{c-1}$ $\quad$ ($|z| > 0$, $\alpha < \arg z < \alpha + 2\pi$).

$$\left(\frac{d}{dz} z^c = \frac{d}{dz} e^{c\log z} = \frac{c}{z} z^c = c \cdot z^{c-1}.\right.$$
$$\left. = \frac{c}{e^{\log z}} e^{c\log z} = c\, e^{(c-1)\log z} = c \cdot z^{c-1}.\right)$$

Define. P.V. $z^c = e^{c\operatorname{Log} z}$: the principle value of z^c

the principle branch of z^c on $D_{-\pi} = \{r>0, -\pi < \theta < \pi\}$

Def. $c \neq 0$, $\quad c^z := e^{z\log c}$: the exponential function with base c.

Fix a value of $\log c$, c^z is an entire function.

Note $\frac{d}{dz} c^z = \frac{d}{dz} e^{z\log c} = \log c \cdot e^{z\log c} = c^z \log c$.

$\Rightarrow c^z \in Hol(\mathbb{C})$.

34. Trigonometric functions

$$e^{ix} = \cos x + i\sin x \qquad\qquad \cos x = \frac{e^{ix}+e^{-ix}}{2}$$
$$e^{-ix} = \cos x - i\sin x \quad \Rightarrow \quad \sin x = \frac{e^{ix}-e^{-ix}}{2i} \quad \forall x \in \mathbb{R}.$$

Define

$$\cos z := \frac{e^{iz}+e^{-iz}}{2}$$

$$\sin z := \frac{e^{iz}-e^{-iz}}{2i}$$

Note $\cos z \Big|_{z = x \in \mathbb{R}} = \cos x.$

$\sin z \Big|_{z=x\in\mathbb{R}} = \sin x.$

Properties:

1. $\sin z, \cos z$ are entire ($\because$ e^{iz} is entire)
2. $\frac{d}{dz}\sin z = \cos z$, $\frac{d}{dz}\cos z = -\sin z$
3. $\sin(-z) = -\sin z$, $\cos(-z) = \cos z$
4. $e^{iz} = \cos z + i\sin z$
5. $\sin(z_1+z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$ $= \frac{e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}}{2i}$
6. $\cos(z_1+z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$ $= \frac{e^{i(z_1+z_2)} + e^{-i(z_1+z_2)}}{2i}$
7. $\sin 2z = 2\sin z\cos z$, $\cos 2z = \cos^2 z - \sin^2 z$
8. $\sin(z+\frac{\pi}{2}) = \cos z$, $\sin(z-\frac{\pi}{2}) = -\cos z$
9. $\sin^2 z + \cos^2 z = 1$
10. $\sin(z+2\pi) = \sin z$; $\sin(z+\pi) = -\sin z$, $\cos(z+\pi) = -\cos z.$
11. $\cos(z+2\pi) = \cos z$
12. $\sin(iy) = i\sinh y$, $\sinh y = \frac{e^y - e^{-y}}{2}$
    $\cos(iy) = \cosh y$, $\cosh y = \frac{e^y+e^{-y}}{2}$
13. $\sin z = \sin x\cosh y + i\cos x\sinh y$
14. $\cos z = \cos x\cosh y - i\sin x\sinh y$
15. $|\sin z|^2 = \sin^2 x + \sinh^2 y$ $\leftarrow$ $\sin z, \cos z$ are not bounded
16. $|\cos z|^2 = \cos^2 x + \sinh^2 y.$

Note: $|\cos x| \leq 1 \quad \forall x \in \mathbb{R}$ but $|\cos z| \nleq 1$

$$\cos(ni) = \frac{e^{i(ni)} + e^{-i(ni)}}{2} = \frac{e^{-n} + e^{n}}{2} = \cosh(n) \to \infty$$

as $n \to \infty$.

(17) $\sin z = 0 \iff z = n\pi$

(18) $\cos z = 0 \iff z = \frac{\pi}{2} + n\pi$

(19) $\tan z := \frac{\sin z}{\cos z}$, $\cot z = \frac{\cos z}{\sin z}$

(20) $\sec z := \frac{1}{\cos z}$, $\csc z := \frac{1}{\sin z}$

(21) $\frac{d}{dz} \tan z = \sec^2 z$, $\frac{d}{dz} \cot z = -\csc^2 z$

(22) $\frac{d}{dz} \sec z = \sec z \tan z$; $\frac{d}{dz} \csc z = -\csc \cot(z)$.

(23) $\tan(z + \pi) = \tan z$.

Example $\tan(i) = \frac{\sin i}{\cos i} =$

35. Hyperbolic functions

$$\sinh z := \frac{e^{z} - e^{-z}}{2}, \quad \cosh z := \frac{e^{z} + e^{-z}}{2}$$

Note $\frac{d}{dz} \sinh z = \cosh z$, $\frac{d}{dz} \cosh z = \sinh z$.

36. Inverse trigonometric and hyperbolic functions

$w = \sin^{-1} z$: the inverse sine function

$\Updownarrow$

$\sin w = z \iff \dfrac{e^{iw} - e^{-iw}}{2i} = z$

$\iff (e^{iw})^2 - 2iz(e^{iw}) - 1 = 0$

$\iff e^{iw} = iz + \sqrt{1-z^2}$

$\iff iw = \log[iz + \sqrt{1-z^2}]$.

$\iff w = -i\log[iz + \sqrt{1-z^2}]$.

Thus

- $\sin^{-1} z = -i\log[iz + \sqrt{1-z^2}]$
- $\cos^{-1} z = -i\log[z + i\sqrt{1-z^2}]$ (Exercise)
- $\tan^{-1} z = \dfrac{i}{2}\log\dfrac{i+z}{i-z}$ (Exercise)

Properties

- $\dfrac{d}{dz}\sin^{-1} z = \dfrac{1}{\sqrt{1-z^2}}$, $\dfrac{d}{dz}\cos^{-1} z = -\dfrac{1}{\sqrt{1-z^2}}$
- $\dfrac{d}{dz}\tan^{-1} z = \dfrac{1}{1+z^2}$.

- $\sinh^{-1} z = \log[z + \sqrt{1+z^2}]$
- $\cosh^{-1} z = \log[z + \sqrt{z^2-1}]$
- $\tanh^{-1} z = \dfrac{1}{2}\log\dfrac{1+z}{1-z}$.

(Ex) 1) $\tan^{-1}(2i)$

2) $\sin z = 2$.

Chapter 4. Integrals

37. Derivatives of functions $w(t)$

$u(t), v(t) : [a, b] \longrightarrow \mathbb{R}$

$w(t) := u(t) + i v(t) \quad : [a, b] \longrightarrow \mathbb{C} \ (= \mathbb{R} + i\mathbb{R})$

$w'(t) := \frac{d}{dt} w(t) = u'(t) + i v'(t)$ if $\exists\ u'(t)$ & $\exists\ v'(t)$

Properties

1) $[z_0 w(t)]' = z_0 w'(t)$

Proof. $z_0 = x_0 + i y_0, \quad w(t) = u(t) + i v(t)$

$z_0 w(t) = x_0 u(t) - y_0 v(t) + i(y_0 u(t) + x_0 v(t))$

$[z_0 w(t)]' = x_0 u'(t) - y_0 v'(t) + i(y_0 u'(t) + x_0 v'(t))$

$= z_0 w'(t)$.

2) $\left(e^{z_0 t}\right)' = z_0 e^{z_0 t}$.

$e^{z_0 t} = e^{x_0 t} \cdot e^{i y_0 t}$

$= e^{x_0 t} \cos y_0 t + i e^{x_0 t} \sin y_0 t$

$\left(e^{z_0 t}\right)' =$

$= z_0 e^{z_0 t}$.

<u>Note</u>: The mean value theorem for $w(t)$ is Not true.

$\left[\; \exists c \in [a,b] \text{ s.t. } w'(c) = \dfrac{w(b) - w(a)}{b-a} \; \right]$

Counter-example: $\quad w(t) = e^{it}, \quad 0 \le t \le 2\pi$

$$w'(c) = ie^{ic} \neq 0 = \frac{w(2\pi) - w(0)}{2\pi - 0} = \frac{0-0}{2\pi} = 0. \quad \boxtimes$$

38. Definite integrals of functions $w(t)$

$$w(t) = u(t) + iv(t), \quad t \in [a,b]$$

$$\int_a^b w(t)\,dt := \overset{\exists}{\int_a^b} u(t)\,dt + i \overset{\exists}{\int_a^b} v(t)\,dt$$

Note 1) $\operatorname{Re} \int_a^b w(t)\,dt = \int_a^b u(t)\,dt = \int_a^b \operatorname{Re} w(t)\,dt$

$\operatorname{Im} \int_a^b w(t)\,dt = \int_a^b v(t)\,dt = \int_a^b \operatorname{Im} w(t)\,dt.$

2) If $u, v : [a,b] \to \mathbb{R}$ are piece-wise continuous then $\exists \int_a^b u\,dt$ & $\int_a^b v(t) \Rightarrow \exists \int_a^b w(t)\,dt.$

Ex. $$\int_0^{\pi/2} e^{it}\,dt = \int_0^{\pi/2} \cos t\,dt + i \int_0^{\pi/2} \sin t\,dt$$
$$= \sin t \Big|_0^{\pi/2} + i(-\cos t)\Big|_0^{\pi/2}$$
$$= 1 + i \left(= \frac{e^{it}}{i}\Big|_0^{\pi/2} = \frac{i-1}{i} = \right.$$

. The fundamental theorem of calculus holds

$$f(t) = u(t) + iv(t), \quad F(t) = U(t) + iV(t)$$

If $F'(t) = f(t) \; \forall t \in [a,b]$, then

$$\int_a^b f(t)\,dt = F(b) - F(a) = F(t)\Big|_a^b.$$

Ex. $$\int_0^{\pi/2} e^{it}\,dt = \frac{e^{it}}{i}\Big|_0^{\pi/2} = 1 + i.$$

- $\int_a^b w(t)\,dt = \int_a^c w(t)\,dt + \int_c^b w(t)\,dt$.

- But it is Not true that

$\exists\, c \in (a,b)$ s.t. $\int_a^b w(t)\,dt = w(c)\,(b-a)$

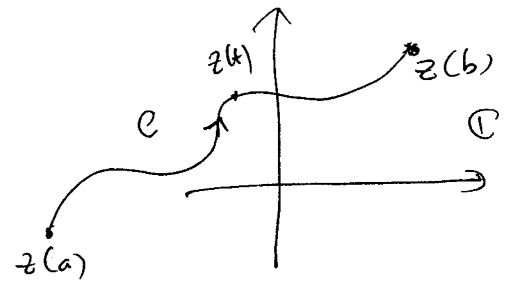
<u>Counterexample</u>

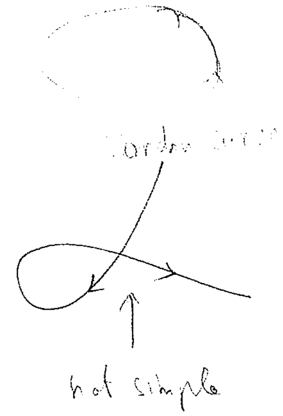
$w(t) = e^{it}$, $\int_0^{2\pi} e^{it}\,dt = \frac{e^{it}}{i}\Big|_0^{2\pi} = 0$

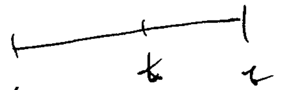
$w(c)\cdot(2\pi - 0) = e^{ic}(2\pi - 0) = 2\pi e^{ic} \neq$

<u>Ex</u>. $\int_0^{\pi} e^{2n\pi i t}\,dt = \begin{cases} 1 & \text{if } n=-1 \\ 0 & \text{if otherwise} \end{cases}$

39. <u>Contours</u>







<u>Def</u> i) an arc (C) is a set of points $z(t) = x(t) + iy(t) = (x(t), y(t))$, $\forall\, a \le t \le b$, where
(a curve)
$x, y : [a,b] \to \mathbb{R}$ are continuous.

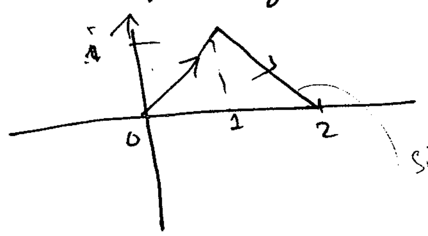
ii) C is simple if $z(t_1) \neq z(t_2)$ $\forall\, t_1 \neq t_2 \in [a,b]$

iii) C is closed if $z(a) = z(b)$

iv) if C is simple and closed, then C is a simple closed curve or <u>a Jordan curve</u>.

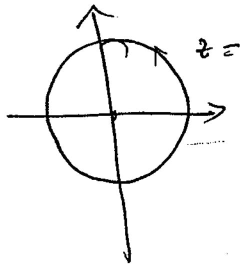
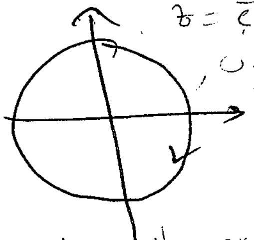
v) C is positively oriented if it is in the counterclockwise direction.

<u>Ex</u>.



$z(t) = \begin{cases} 1 + it & \text{if } 0 \le t \le 1 \\ t + (2-t)i & \text{if } 1 \le t \le 2 \end{cases}$

simple / not closed

2)  $z = e^{it}$ $0 < t \le 2\pi$ → a Jordan curve positively-oriented direction ←  $z = \bar{e}^{it}$, $0 < t \le 2\pi$ not posit.-oriented

Note: The parametric representation for any arc is Not unique

Consider $\phi : [\alpha, \beta] \xrightarrow{1-1} [a, b]$, $\phi \in C^1$; $\phi'(\tau) > 0$ $\forall \tau \in [\alpha, \beta]$.

$\tau \longmapsto t = \phi(\tau)$, $dt = \phi'(\tau) d\tau$

$$C : \quad z = z(t) = x(t) + i y(t), \quad t \in [a, b]$$

$$\Leftrightarrow C : \quad z = \tilde{z}(\tau), \quad \tilde{z}(\tau) = z[\phi(\tau)], \quad \tau \in [\alpha, \beta].$$

Suppose that $C : \quad z = z(t) = x(t) + i y(t), \quad t \in [a, b]$

& $\exists\, x'(t), y'(t)$ are continuous on $[a, b]$.

Then $|z'(t)| = \sqrt{x'(t)^2 + y'(t)^2}$

$$L := \int_a^b |z'(t)| dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt$$: the length of C.

Note: L is invariant under the change $t = \phi(\tau)$.

Proof
$$\tilde{L} = \int_\alpha^\beta |\tilde{z}'(\tau)| d\tau = \int_\alpha^\beta |z'(\phi(\tau))\, \phi'(\tau)| d\tau$$

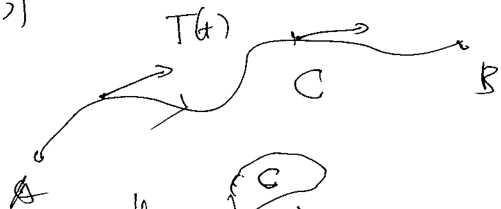
$$= \int_\alpha^\beta |z'(\phi(\tau))|\, |\phi'(\tau)| d\tau$$

$$= \int_\alpha^\beta |z'(\phi(\tau))|\, \phi'(\tau) d\tau = \int_\alpha^\beta |z'(\phi(\tau))|\, dt$$

by Calculus $$\int_a^b |z'(t)|\, dt = L.$$

Def 1) C is smooth if $\exists\, z'(t) \neq 0$ $\forall\, a \leq t \leq b$.

$$T(t) := \frac{z'(t)}{|z'(t)|} \quad : \text{the unit tangent vector.}$$



2) A contour is a piecewise smooth arc, i.e.,

$C = C_1 \cup C_2 \cup \cdots \cup C_n$, C_j is smooth $1 \leq j \leq n$,

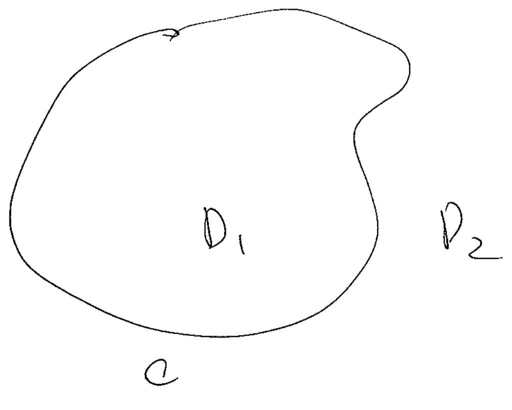
C_j : joined end to end.

$(C_j \cap C_{j+1} = \{A_j\})$.

__Jordan curve theorem__. Let C be a simple closed contour.

Then $\exists\, D_1, D_2$ are domain in $\mathbb{C}$ s.t.

$C = \partial D_1 = \partial D_2$, D_1 is bounded
D_2 is unbounded
$D_1 \cap D_2 = \emptyset$.



40 Contour integrals

Suppose that

- C : a contour, given by $z = z(t)$, $a \le t \le b$
  $z_1 = z(a)$
  $z_2 = z(b)$

- $f : C \longrightarrow \mathbb{C}$ is piecewise continuous, i.e., $f(z(t))$ is piecewise continuous on $[a, b]$.

Define
$$\int_C f(z)\,dz := \int_a^b f(z(t))\, z'(t)\,dt$$
: the line integral (contour integral) of f along C.

Note $\int_C f(z)\,dz$ is invariant under ~~the~~ a changes in the representation of C.

Proof. $t = \phi(\tau)$, $\phi : [\alpha, \beta] \xrightarrow{1-1} [a, b]$, $\phi'(\tau) > 0$ $\forall\, \tau \in (\alpha, \beta)$.

$$\int_a^b f(z(t))\, z'(t)\,dt = \int_\alpha^\beta f(z(\phi(\tau))) \cdot z'(\phi(\tau))\, \phi'(\tau)\,d\tau \qquad y(\tau) = z(\phi(\tau))$$
$$= \int_\alpha^\beta \cancel{f(y(\tau))\, y'(\tau)\, d\tau}$$
$$= \int_\alpha^\beta f(y(\tau))\, y'(\tau)\,d\tau \qquad \Box$$

$C: \; z = z(t) \quad a \le t \le b$

$C: \; z = y(\tau) = z(\phi(\tau)), \; \alpha \le \tau \le \beta$

Properties:

① $$\int_C z_0 f(z)\,dz = z_0 \int_C f(z)\,dz \ , \quad z_0 \in \mathbb{C}$$

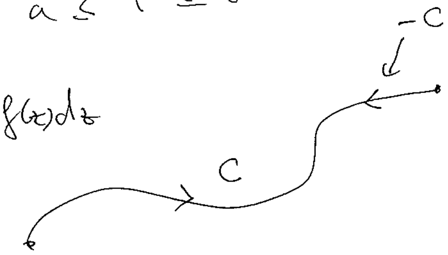
② $$\int_C \big(f(z) \pm g(z)\big)\,dz = \int_C f(z)\,dz \pm \int_C g(z)\,dz$$

③ $$\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$$

$-C$: $z = z(-t)$, $-b \le t \le -a$ represents $-C$

C : $z = z(t)$ $\qquad a \le t \le b$ $\qquad\qquad C$

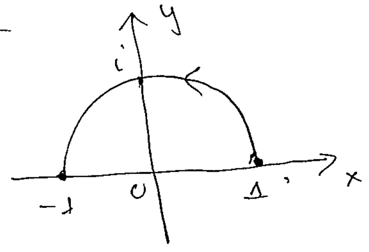
④ $$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$



$C = C_1 + C_2$.

41. Some examples

a) $C : z = e^{i\theta}$, $0 \le \theta \le \pi$



- $$\int_C \bar{z}\,dz = \int_0^{\pi} e^{-i\theta} . i e^{i\theta}\,d\theta = \int_0^{\pi} i\,d\theta = 2\pi i$$

- $$\int_C z\,dz = \int_0^{\pi} e^{i\theta}\, i e^{i\theta}\,d\theta = i \int_0^{\pi} e^{2i\theta}\,d\theta = i \frac{e^{2i\theta}}{2i}\Big|_0^{\pi} = 0$$

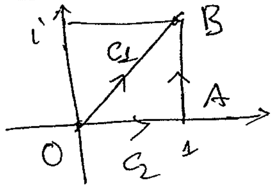
- $$\int_C \frac{1}{z}\,dz = \int_0^{\pi} e^{-i\theta}\, i e^{i\theta}\,d\theta = 2\pi i .$$

b) $C : z = e^{i\theta}$, $0 \le \theta \le 2\pi$

- $$\int_C z^n\,dz = \begin{cases} 2\pi i & \text{if } n = -1 \\ 0 & \text{if otherwise} \end{cases} \qquad n \in \mathbb{Z}$$

. In general $$\int_{|z| = R > 0} z^n\,dz = \begin{cases} 2\pi i & \text{if } n = -1 \\ 0 & \text{if otherwise.} \end{cases}$$

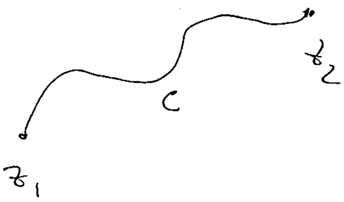
c) C_1 :



$$\int_{C_1} \bar{z}\, dz = \int_0^1 \overline{1+it}\; i\,dt = \int_0^1 (1-it)\, i\,dt = i + \frac{1}{2}.$$

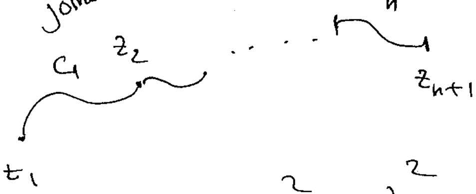
$$\int_{C_2} \bar{z}\, dz = \int_0^1 \overline{t+i0}\, dt + \int_0^1 \overline{it}\; i\,dt = \int_0^1 t\,dt + \int_0^1 t\,dt = 1.$$

d) i) $$\int_C z\, dz = \int_a^b z(t).z'(t)\, dt$$



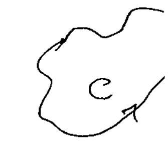
$$= \int_a^b \frac{d}{dt}\frac{[z(t)]^2}{2} = \frac{z^2(t)}{2}\Big|_a^b = \frac{z^2(b) - z^2(a)}{2}$$

$$= \frac{z_2^2 - z_1^2}{2}$$



ii) $C = C_1 + C_2 + \dots + C_n$

$$\int_C z\,dz = \sum_{k=1}^n \int_{C_k} z\,dz = \sum_{k=1}^n \frac{z_{k+1}^2 - z_k^2}{2} = \frac{z_{n+1}^2 - z_1^2}{2}.$$



Remark 1) $\oint_C z\,dz = 0$. $\quad \oint_C P_n(z)\,dz = 0$.

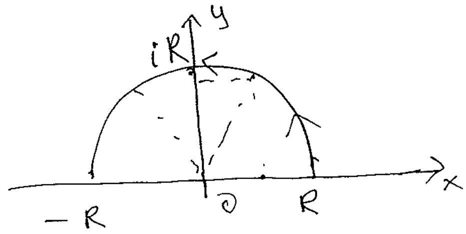
$C \leftarrow$ closed

2) in general $\oint_C f(z)\,dz = 0$ for $f \in Hol(D)$, $C \subset D$.

$C \leftarrow$ closed

42. Examples with branch cuts

a) <u>Example 1</u>.

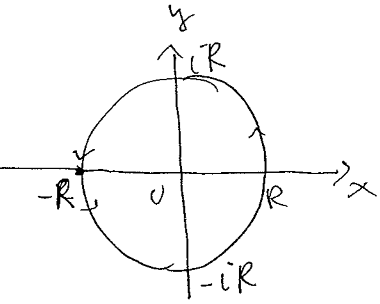


$$C: \quad z = Re^{it} \qquad 0 \le t \le \pi$$
$$dz = iRe^{it}dt$$

$$f(z) = \sqrt[3]{z}$$
$$= \sqrt[3]{r}\, e^{i\frac{\theta}{3}}, \quad z = re^{i\theta}, \quad r>0, \quad 0<\theta<2\pi$$

$$\int_C f(z)\,dz = \int_0^{\pi} \sqrt[3]{R}\, e^{i\frac{t}{3}} . Rie^{it}\,dt = i\int_0^{\pi} R\sqrt[3]{R} . e^{i\frac{4t}{3}}\,dt$$
$$= iR\sqrt[3]{R} . \frac{e^{i\frac{4t}{3}}}{i\frac{4}{3}}\Bigg|_0^{\pi} = \frac{3}{4}R\sqrt[3]{R}\left(e^{i\frac{4}{3}\pi} - 1\right)$$
$$= -\frac{3}{4}R\sqrt[3]{R}\left(e^{i\pi/3} + 1\right) = -\frac{3}{4}R\sqrt[3]{R}\left(\frac{3}{2} + \frac{\sqrt{3}}{2}i\right).$$

b) <u>Example 2</u>.



$$C: \quad z = Re^{it} \qquad -\pi \le t \le \pi$$

$$f(z) = z^{a} = e^{a\log z} \qquad (a \ne -1)$$
$$= e^{a(\log|z| + i\operatorname{Arg} z)}$$

$$\int_C f(z)\,dz = \int_{-\pi}^{\pi} e^{a(\ln R + it)} . Rie^{it}\,dt$$
$$= \int_{-\pi}^{\pi} R^a . iR . e^{i(a+1)t}\,dt$$
$$= iR^{a+1}\, \frac{e^{i(a+1)t}}{i(a+1)}\Bigg|_{-\pi}^{\pi} = 2i\,\frac{R^{a+1}}{a+1} . \frac{e^{i(a+1)\pi} - e^{-i(a+1)\pi}}{2i}$$
$$= \frac{2iR^{a+1}}{a+1}\sin(a+1)\pi.$$

<u>Note</u>.
$$\int_C z^n\,dz = \begin{cases} 2\pi i & \text{if } n=-1 \\ 0 & \text{if } n \ne -1 \end{cases} \qquad n \in \mathbb{Z}.$$

43. Upper bounds for moduli of contour integrals

Lemma. $w : [a,b] \longrightarrow \mathbb{C}$ is a piecewise continuous. Then

$$\left|\int_a^b w(t)\,dt\right| \le \int_a^b |w(t)|\,dt \quad (1)$$

Proof

⊕ $\int_a^b w(t)\,dt = 0 \Rightarrow$ (1) holds.

⊕ $0 \neq \int_a^b w(t)\,dt = R.e^{i\theta_0}$.

$$R = \left|\int_a^b w(t)\,dt\right| = \left|e^{-i\theta_0}\int_a^b w(t)\,dt\right| = \underbrace{\int_a^b e^{-i\theta_0} w(t)\,dt}_{\in \mathbb{R}} =$$

$$= \mathrm{Re}\int_a^b e^{-i\theta_0} w(t)\,dt = \int_a^b \mathrm{Re}\left[e^{-i\theta_0} w(t)\right] dt$$

$$\le \int_a^b \left|e^{-i\theta_0} w(t)\right| dt = \int_a^b |w(t)|\,dt.$$

Theorem. C is a contour of length L.

$f : C \longrightarrow \mathbb{C}$ is piecewise continuous,

$|f(z)| \le M \quad \forall z \in C$.

Then $\left|\int_C f(z)\,dz\right| \le M.L$

Proof

$$\left|\int_C f(z)\,dz\right| = \left|\int_a^b f(z(t))\, z'(t)\,dt\right| \overset{\text{by Lemma}}{\le} \int_a^b |f(z(t))\, z'(t)|\,dt.$$

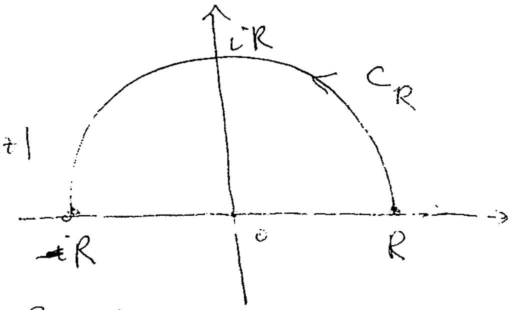
$$\le \int_a^b M.|z'(t)|\,dt = M.\int_a^b |z'(t)|\,dt = M.L.$$

Note: if $f : C \to \mathbb{C}$ is piecewise continuous, then

$\exists M := \max_{z \in C} |f(z)|$ s.t. $|f(z)| \le M \ \forall z \in C$.

(if $f : [a,b] \longrightarrow \mathbb{R}$ is continuous, then f is bounded on $[a,b]$).

Example



1) $\left| \int_{C_R} \frac{dz}{z^4+1} \right| \leq \int_{C_R} \left| \frac{1}{z^4+1} \right| |dz|$

$\leq \frac{1}{R^4-1} \cdot \pi R \longrightarrow 0$ as $R \to +\infty$.

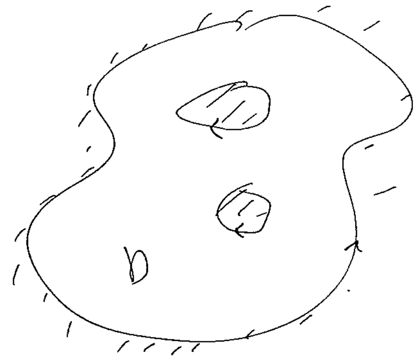
2) $\left| \int_{C_R} \frac{z^2+1}{z^4+z^2+1} \, dz \right| \leq \frac{R^2+1}{R^4-R^2-1} \cdot \pi R \longrightarrow 0$ as $R \to +\infty$

3) $\left| \int_{C_R} \frac{\operatorname{Log} z}{z^4+1} \, dz \right| \leq \frac{\sqrt{\ln^2 R + \pi^2}}{R^4-1} \cdot \pi R$

(L ... $\operatorname{Log} R$ ... $\sqrt{\ln R ...}$

44-45. Anti-derivatives

. Let $D \subset \mathbb{C}$ be a domain



. $f : D \to \mathbb{C}$ is continuous

Def. An anti-derivatives $F(z)$ of $f(z)$

is $F : D \to \mathbb{C}$ satisfying $\exists$ $F'(z) = f(z)$ $\forall z \in D$

$F \in \mathrm{Hol}(D)$ ($F \in$

Remark. an anti-derivative of $f(z)$ is unique except an additive constant

i.e., if F_1 & F_2 are ~~an~~ anti-derivatives of f then $\exists c \in \mathbb{C}$

s.t $F_1(z) - F_2(z) \equiv c$ $\forall z \in D$

Proof $h(z) := F_1(z) - F_2(z) \in \mathrm{Hol}(D)$

$h'(z) = F_1'(z) - F_2'(z) = 0$ $\forall z \in D$

$\hookrightarrow h(z) = \text{const}$. $\boxtimes$

Example 1) $F(z) = z^2/2$, $f(z) = z \Rightarrow F'(z) = f(z)$

$\hookrightarrow$ $z^2/2$ is an anti-derivative of z.

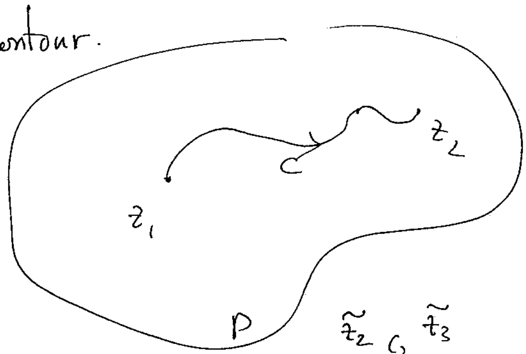
Theorem. Suppose that $f: \mathbb{C} \supset D \to \mathbb{C}$ is continuous. Then the following are equivalent:

a) $\exists$ an anti-derivative $F(z)$ of $f(z)$ on D.

b) $$\int_C f(z)\,dz = \int_{z_1}^{z_2} f(z)\,dz = F(z)\Big|_{z_1}^{z_2} = F(z_2) - F(z_2)$$

, where C is a contour $\subset D$ extending from z_1 to z_2

c) $$\oint_C f(z)\,dz = 0 \text{ for any closed contour.}$$

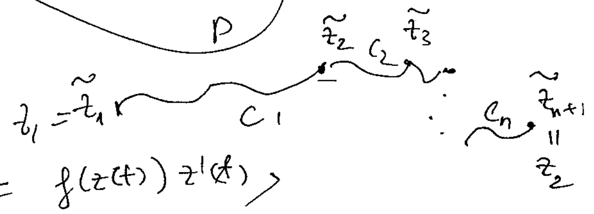


Proof.

• a) $\Rightarrow$ b)

+) $C: \; z = z(t), \quad a \le t \le b$

$z_1 = z(a), \; z_2 = z(b)$



Since $\frac{d}{dt} F(z(t)) = F'(z(t)).z'(t) = f(z(t))\,z'(t)$,

$$\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt = \int_a^b \frac{d}{dt} F(z(t)) = F(z(t))\Big|_{t=a}^{b} = F(z_2) - F(z_1)$$

+) $C = C_1 + \dots + C_n \Rightarrow \int_C f(z)\,dz = \sum \int_{C_j} f(z)\,dz = \sum_{j=1}^{n} \left(F(\tilde{z}_{j+1}) - F(\tilde{z}_j)\right)$

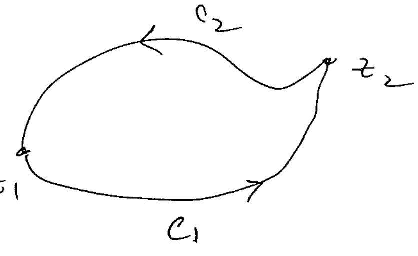
$= F(\tilde{z}_{n+1}) - F(\tilde{z}_1)$

$= F(z_2) - F(z_1)$

• b) $\Rightarrow$ c)

Let C be a closed contour.

Choose $z_1, z_2 \in C$. Then



$$C = C_1 - C_2.$$

Thus
$$\oint_C f\,dz = \int_{C_1} f(z)\,dz + \int_{-C_2} f(z)\,dz = F(z_2) - F(z_1) - \left(F(z_2) - F(z_1)\right) = 0$$

• c) $\Rightarrow$ a)

Fix $z_0 \in D$, define

$$F(z) := \int_{z_0}^{z} f(z)\,dz, \quad z \in D$$

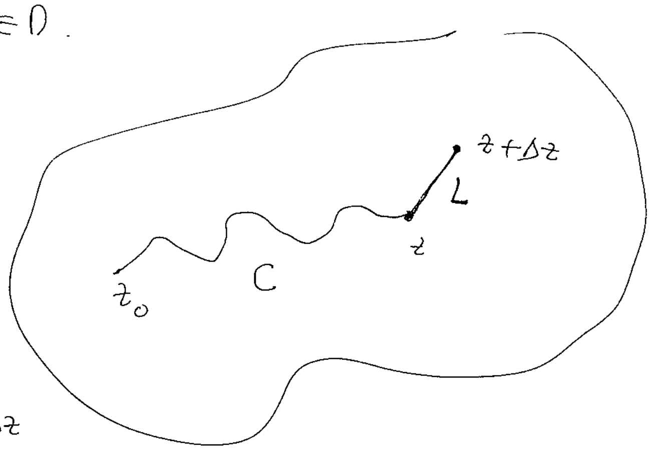
+ F is well-defined because if

C_1 : a contour from z_0 to z

C_2 : a ________ z_0 to z

Then $\displaystyle\int_{C_1 - C_2} f(z)\,dz \overset{\text{by } ©}{=} 0 \Rightarrow \int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$.

+: $\exists\ F'(z) = f(z)\ \forall z \in D$.



$F(z+\Delta z) - F(z) =$

$$= \int_{z_0}^{z+\Delta z} f(z)dz - \int_{z_0}^{z} f(z)dz$$

$$= \int_{C+L} f(z)dz - \int_{C} f\,dz = \int_{z}^{z+\Delta z} f(z)\,dz$$

$$\Rightarrow \frac{F(z+\Delta z) - F(z)}{\Delta z} = \frac{1}{\Delta z} \cdot \int_{z}^{z+\Delta z} f(z)\,dz$$

$$\Rightarrow \left| \frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) \right| = \left| \frac{1}{\Delta z} \int_{z}^{z+\Delta z} (f(s) - f(z))\,ds \right|$$

$$\leq \max_{s \in L} |f(s) - f(z)| \cdot \frac{|\Delta z|}{|\Delta z|} = \max_{s \in L} |f(s) - f(z)| \to 0$$

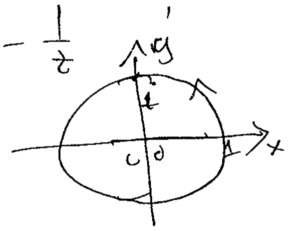
as $\Delta z \to 0$

$(\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $|f(s) - f(z)| < \varepsilon\ \forall\ |s - z| < \delta)$

Thus $\exists\ F'(z) = \lim_{\Delta z \to 0} \frac{F(z+\Delta z) - F(z)}{\Delta z} = f(z)$ ☒

Examples

1) $f(z) = \frac{1}{z^2}$ has an anti-derivative $F(z) = -\frac{1}{z}$



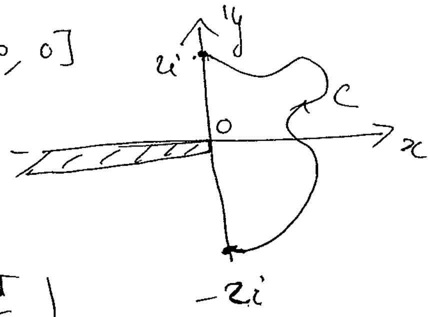
$$\Rightarrow \int_{|z|=1} \frac{dz}{z^2} = 0 .$$

2) $f(z) = \frac{1}{z}$ has no anti-derivative on $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ since $\int_{|z|=1} \frac{dz}{z} = 2\pi i \neq 0.$

but a) $f(z) = \frac{1}{z}$ has an anti-derivative on $\mathbb{C} \setminus [-\infty, 0]$

$$F(z) = \mathrm{Log}\, z = \ln|z| + i\theta , \quad -\pi < \theta < \pi$$

$$F'(z) = \frac{1}{z} = f(z) \quad \forall z \in \mathbb{C} \setminus [-\infty, 0]$$

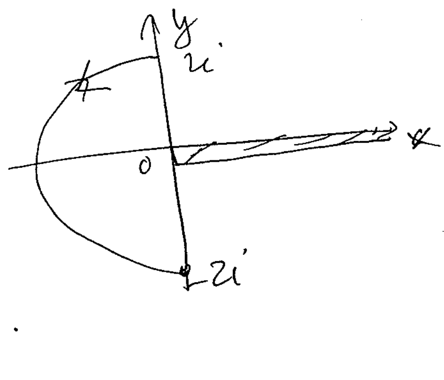


$$\int_{-2i}^{2i} \frac{dz}{z} = \mathrm{Log}(2i) - \mathrm{Log}(-2i)$$

$$= \ln 2 + i\pi/2 - \left(\log 2 - \frac{i\pi}{2}\right)$$

$$= i\pi$$

b) $F(z) = \mathrm{Log}\, z = \log|z| + i\theta \quad 0 < \theta < 2\pi$ on $\mathbb{C} \setminus \mathbb{R}^+ = \mathbb{C} \setminus [0, +\infty)$ is an anti-derivative of $\frac{1}{z}$



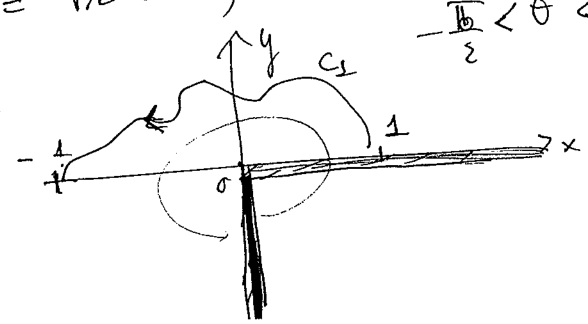
$$\int_{+2i}^{-2i} \frac{dz}{z} = \mathrm{Log}\, 2i - \mathrm{Log}(-2i)$$

$$= \frac{3\pi i}{2} - \frac{\pi i}{2} = \pi i$$

So $$\int_{|z|=2} \frac{dz}{z} = \pi i + \pi i = 2\pi i .$$

---

3) $f_1(z) = z^{1/3} = \sqrt[3]{z} = \sqrt[3]{r}\, e^{i\theta/3}$, $z = re^{i\theta}$, $r > 0$, $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$



$$F_1(z) = \frac{3}{4} z^{4/3} = \frac{3}{4} r^{4/3} e^{i4\theta/3}, \quad -\frac{\pi}{2} < \theta < \frac{3\pi}{2}$$

$$F_1'(z) \overset{\text{exercise}}{=} f(z)$$

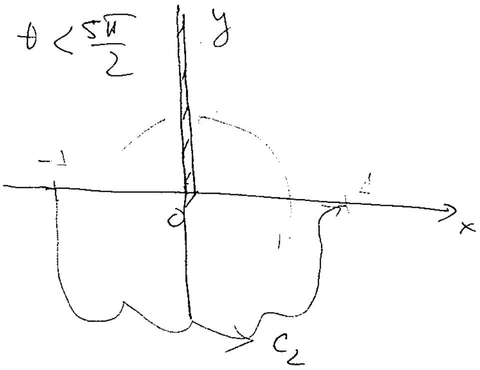
$$\hookrightarrow \int_{C_1} \sqrt[3]{z}\, dz = F_1(-1) - F(1) = \frac{3}{4}\left(e^{i\frac{4\pi}{3}} - e^{i0}\right)$$

$$= -\frac{3}{4}\left(e^{i\pi/3} + 1\right) = -\frac{3}{4}\left(\frac{3}{2} + \frac{\sqrt{3}}{2} i\right)$$

• $$f_2(z) = z^{1/3} = \sqrt[3]{r}\, e^{i\theta/3}, \quad \frac{\pi}{2} < \theta < \frac{5\pi}{2}$$

$$F_2(z) = \frac{3}{4} z^{4/3} = \frac{3}{4} r^{4/3} e^{i4\theta/3}$$

$$\frac{\pi}{2} < \theta < \frac{5\pi}{3}$$



is an anti-derivative of $f_2(z)$.

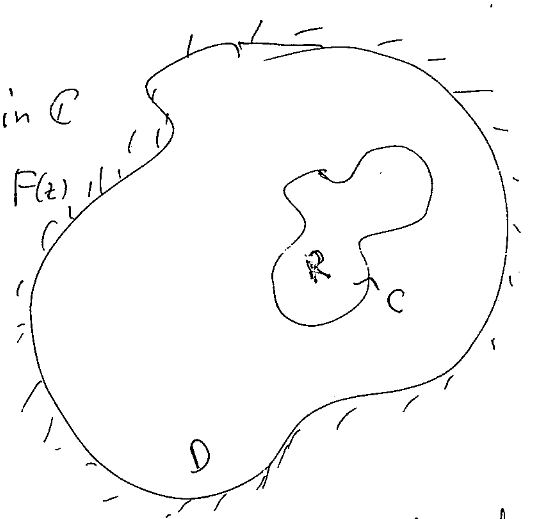
$$\int_{C_2} f_2(z)\, dz = F_2(1) - F_2(-1)$$

$$= \frac{3}{4}\left(e^{i8\pi/3} - e^{i4\pi/3}\right)$$

$$= \frac{3}{4}\left(e^{2\pi i/3} + e^{\pi i/3}\right) = 2i\,\frac{3}{4}\,\frac{e^{\pi i/3} - e^{-\pi i/3}}{2i}$$

$$= \frac{3}{2} i \sin \pi/3 = \frac{3\sqrt{3}}{4} i \quad \boxtimes$$

46. <u>Cauchy - Goursat theorem</u>



<u>Recall.</u> Let $D \subset \mathbb{C}$ be a domain in $\mathbb{C}$

if $f \in C^0(D)$ has an anti-derivative $F(z)$, then

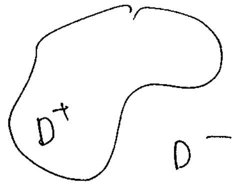
$$\int_{C \leftarrow \text{closed contour}} f(z)\,dz = 0$$

<u>Question.</u> $\exists$ other conditions on f which ensure that $\int_{C \leftarrow \text{closed contour}} f(z)\,dz = 0$

<u>Answer.</u> If $f \in Hol(D)$, then $\int_C f(z)\,dz = 0$.
(Cauchy - Goursat's theorem)

Let C be a simple closed contour

Let $R :=$ the closed region bounded by C



$R = \overline{D^+} = D^+ \cup \partial D$

Theorem (Cauchy - Goursat)
if f is analytic in R, then $\int_{C \leftarrow \text{simple closed contour}} f(z)\,dz = 0$.

<u>Proof</u> (in general, see the proof in text book, Sec. 47)
$\oplus$ f' is continuous on/in R (R is closed region)

Suppose $f \in Hol(R)$ & $f' \in C^0(R)$.

$f(z) = u(x,y) + i v(x,y)$, $C: z(t) = x(t) + i y(t)$, $a \le t \le b$

$z'(t) = x'(t) + i y'(t)$

$$\int_C f(z)\,dz = \int_C (u + iv)(x'(t) + iy'(t))\,dt$$

$$= \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy$$

$$\underline{\underline{\text{by Green's theorem}}} \iint_R (-v_x - u_y)\,dx\,dy + i \iint_R (u_x - v_y)\,dx\,dy = 0$$

$\quad (-v_x - u_y) = 0,\ (u_x - v_y) = 0$ by C-R eq.

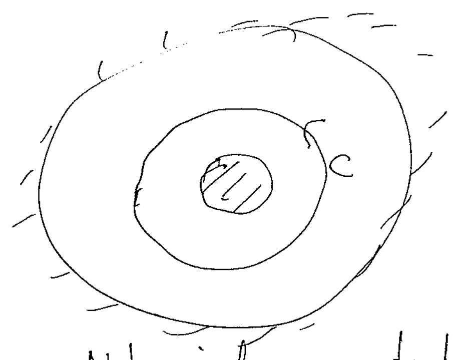
Example $\oint e^{z^3} dz = 0$

4.8. Simply connected domains

Let $D \subset \mathbb{C}$ be a domain

Def. D is simply connected if $\forall$ simple closed contour in D encloses only points of D

$\forall C \subset D \Rightarrow R \subset D.$



Not simply connected

Theorem. if $f \in Hol(D)$ and D is simply connected, then

$$\int_C f(z)dz = 0 \text{ for any closed contour lying in } D:$$

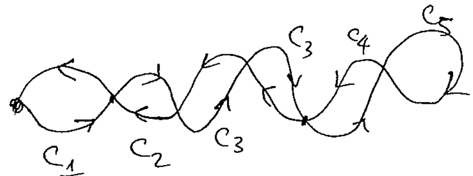
Proof

\+ C is simple, ~~by~~ Cauchy–Goursat's theorem we have $\int_C f(z)dz = 0$

itself

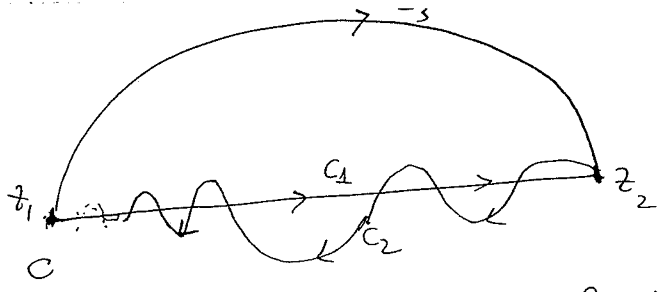
\+ C intersects ~~itself~~ a finite number of times
itself

$C = C_1 + C_2 + \cdots + C_n$



$$\Rightarrow \int_C f\,dz = \sum_{k=1}^{n} \underbrace{\int_{C_k} f(z)dz}_{=0} = 0.$$

\+ C intersects ~~itself~~ a infinite number of times



$C_1 : z_1 \to z_2$

Choose z_1, z_2 and a contour c_3 s.t

$$C = C_1 - C_2$$

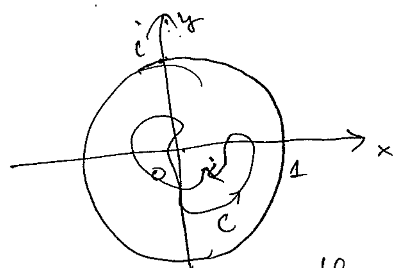
$$\Rightarrow \int_{C_1 - C_3} f(z)\,dz = 0 \quad \text{by the Cauchy - Goursat's theorem}$$

$$\& \int_{C_2 - C_3} f\,dz = 0$$

$$\Rightarrow \int_{C_1} f\,dz = \int_{C_3} f\,dz = \int_{C_2} f\,dz$$

$$\Rightarrow \int_C f dz = \int_{C_1 - C_2} f\,dz = \int_{C_1} f dz - \int_{C_2} f\,dz = 0 \ \boxtimes$$

<u>Example</u>. $\displaystyle\oint_C \frac{dz}{z^{2013}+2} = 0 \quad \forall\, C \subset \{|z|<1\}$.



<u>Corollary</u>. If $f \in Hol(D)$ & D is simply connected, then f has an anti-derivative on D $\left(\exists F \in Hol(D) \text{ s.t. } F'(z) = f(z) \ \forall z \in D\right)$.

proof $\displaystyle\oint_C f dz = 0 \xLeftrightarrow{\text{by theorem in Sec. 44}} \exists$ anti-derivative of f $\boxtimes$

<u>Example</u>. 1) $\displaystyle\oint_C e^{z^{2013}}\,dz = 0$

$$F(z) := \int_0^z e^{s^{2013}}\,ds .$$

2) $\displaystyle\oint_C e^{z^2} z\,dz = 0$, $\quad f(z) = e^{z^2} z$

$$F(z) = \frac{1}{2} e^{z^2} + C .$$

49. Multiply-connected domain

Def. A domain is multiply connected if it is not simply connected

Orientation. Let $D \subset \mathbb{C}$ be a domain.

. positive orientation of ∂D: While you are walking along the boundary ∂D of D if your head is pointing in the same direction as the unit normal vectors while the surface (domain) is on the left then you are walking in the positive direction on ∂D.

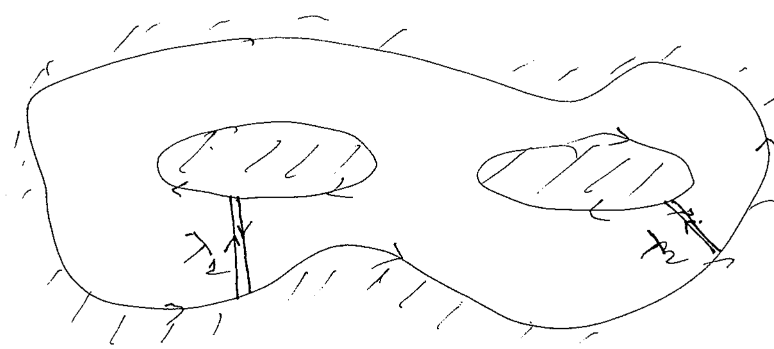
differential form

Recall. $\displaystyle\int_{\partial D} w = \int_{D} dw$ (Stokes' theorem) $\displaystyle\int_{\partial D} P\,dx \ldots$

Cauchy–Goursat's theorem. $\displaystyle\int_{\partial D} f(z)\,dz = 0$ for $f \in Hol(D) \cap C^0(\overline{D})$

$\partial D \leftarrow$ positively oriented.

Proof



for multiply-connected domains

Theorem (Cauchy-Goursat). Suppose that

a) C is a simple closed contour (counterclockwise direction)

b) C_k $(k=1,2,\dots,n)$ are simple closed contours interior to C (clockwise direction)

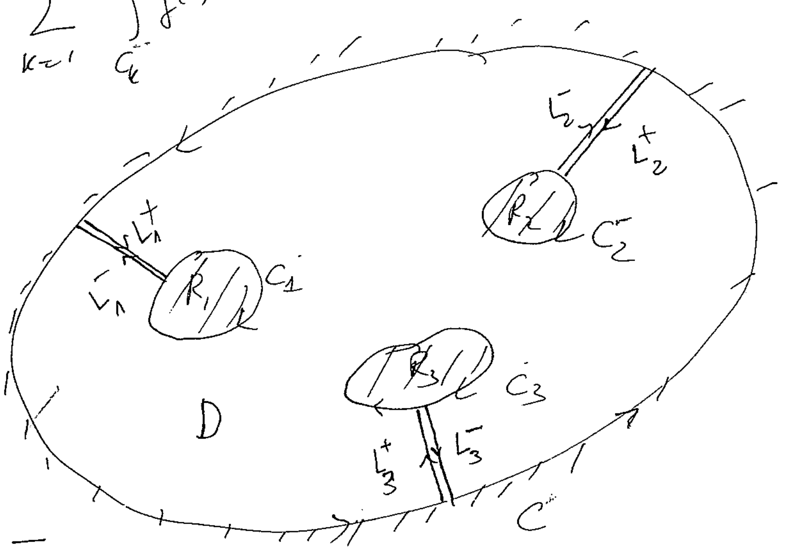
s.t. $C_j \cap C_l = \emptyset$, $R_j \cap R_l = \emptyset$,

where R_k = the interior of C_k, R_0 = the interior of C.

Then if $f \in Hol(\overline{R_0} \setminus \bigcup_{k=1}^{n} R_k)$ then

$$\int_C f(z)\,dz + \sum_{k=1}^{n} \int_{C_k} f(z)\,dz = 0$$

Proof.



$$D := R_0 \setminus \bigcup_{k=1}^{n} \overline{R_k}$$

$$\partial D = C - C_1 - C_2 - \dots - C_n$$

L_k : a path connecting C to C_k, $k = 1, 2, \dots, n$.

$$D^* := D \setminus \bigcup_{k=1}^{n} L_k \Rightarrow D^* \text{ is simply-connected.}$$

$$\partial D^* = \partial D + \sum_{k=1}^{n} (L_k^+ + L_k^-)$$

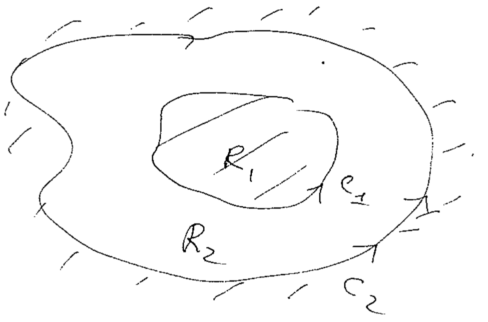
by Cauchy-Goursat's theorem $\int_{\partial D^*} f(z) = 0$

$$\Leftrightarrow \int_{\partial D} f(z)\,dz + \sum_{k=1}^{n} \Big(\underbrace{\int_{L_k^+} f\,dz + \int_{L^-} f\,dz}_{=0} \Big) = 0$$

$$\Leftrightarrow 0 = \int_{\partial D} f\,dz = \int_{C_0} f\,dz + \sum_{k=1}^{n} \int_{C_k} f\,dz = 0 \ \boxtimes$$

Corollary. Let C_1 & C_2 denote positively oriented simple closed contours, where C_1 is interior to C_2. If $f \in Hol(\overline{R}_2 \setminus R_1)$, then

$$\int_{C_2} f(z)\,dz = \int_{C_1} f(z)\,dz .$$



Proof. $D = R_2 \setminus R_1$

R_j : the interior domain bounded by C_j.

$$\int_{\partial D} f\,dz = 0 \Rightarrow \int_{C_2} f\,dz \cancel{+} \int_{C_1^{+}} f\,dz = 0 \Rightarrow \int_{C_2} f\,dz = \int_{C_1} f\,dz .$$

Example 1)

$$\oint_{|z| = \varepsilon > 0} \frac{dz}{z} = 2\pi i$$

$$\oint_{C} \frac{dz}{z} = \int_{|z| = \varepsilon > 0} \frac{dz}{z} = 2\pi i$$

2)

$$\oint_{C} \frac{dz}{z^n} = \int_{|z| = \varepsilon > 0} \frac{dz}{z^n} = 0 \quad \forall n \neq 1 .$$

50. Cauchy integral formula.

Theorem Let C be a simple closed contour & D be a domain bounded by C. ~~Then~~ Let $f \in Hol(\overline{D})$. Then $\forall z_0 \in D$ we have

$$f(z_0) := \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\,dz = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z - z_0}\,dz$$

the Cauchy integral formula

Example

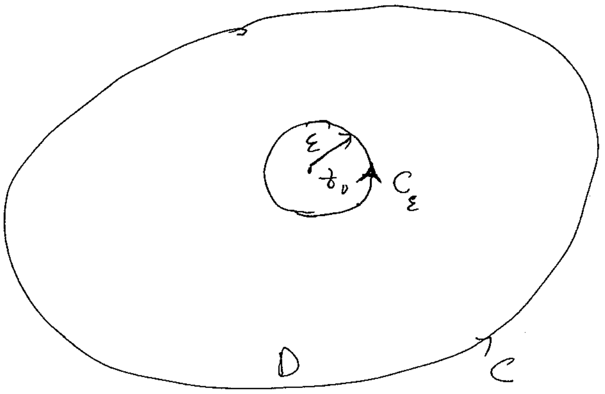
$$\int_{|z| = 2} \frac{\sin z}{z - 1}\,dz = 2\pi i \sin z \Big|_{z = 1} = 2\pi i \sin 1 .$$

Proof of theorem

- $\varepsilon > 0$ small enough.

$D_\varepsilon(z_0) = \{ |z - z_0| < \varepsilon \}$

$C_\varepsilon(z) = \partial D_\varepsilon(z_0) = \{ |z - z_0| = \varepsilon$



- $f \in Hol(\overline{D} \setminus D_\varepsilon(z_0))$

$$\Longrightarrow \quad \frac{1}{2\pi i}\int_C \frac{f\,dz}{z - z_0} = \frac{1}{2\pi i}\int_{C_\varepsilon} \frac{f(z)}{z - z_0}\,dz \overset{?}{=} f(z_0).$$

Indeed,
$$\left| f(z_0) - \frac{1}{2\pi i}\int_{C_\varepsilon} \frac{f(z)}{z - z_0}\,dz \right| = \left| \frac{1}{2\pi i}\int_{C_\varepsilon} \frac{f(z_0) - f(z)}{z - z_0}\,dz \right|$$

$$\leq \frac{1}{2\pi} \int_{C_\varepsilon} \frac{|f(z_0) - f(z)|}{\varepsilon}\,|dz|$$

$$\leq \max_{z \in C_\varepsilon} |f(z_0) - f(z)| \xrightarrow[\varepsilon \to 0]{} 0$$ since f is continuous at z_0 $\boxtimes$

$\left(\forall \varepsilon_0 > 0, \ \exists \varepsilon > 0 \text{ s.t. } |f(z) - f(z_0)| < \varepsilon_0 \ \forall \ |z - z_0| < \varepsilon \right)$

51. An extension of the Cauchy integral formula

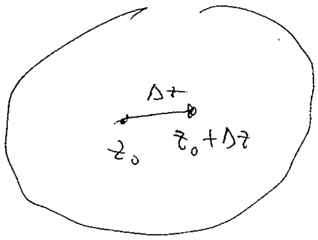
Theorem. Let C be a simple closed contour and D be a bounded domain bounded by C. Let $f \in Hol(\overline{D})$. Then $\forall z_0 \in D$ we have

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\partial D = C} \frac{f(z)}{(z - z_0)^{n+1}}\,dz \quad (n = 0, 1, 2 \ldots)$$

$n = 0$, $\quad f(z_0) = \dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(z)}{z - z_0}\,dz$ (we know).

Proof (by induction on n)

$\boxed{n=1}$ $\quad f'(z_0) = \dfrac{1!}{2\pi i}\displaystyle\int_C \frac{f(z)}{(z-z_0)^2}\,dz.$



$\boxed{n\geq 2}$ (Exercise)

$$\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} = \frac{1}{2\pi i\Delta z}\int_C \frac{f(z)}{(z-z_0-\Delta z)}\,dz - \frac{1}{2\pi i\Delta z}\int_C \frac{f(z)}{(z-z_0)}\,dz$$

$$= \frac{1}{2\pi i\Delta z}\int_C \left[\frac{1}{z-z_0-\Delta z} - \frac{1}{z-z_0}\right] f(z)\,dz =$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)(z-z_0-\Delta z)}\,dz$$

- $d := \min\limits_{z\in C} |z-z_0| = d(z_0,\partial D)$, $\quad L = $ length of C
- $M>0$ s.t. $\quad |f(z)| \leq M \ \forall z\in C$.

$$\Rightarrow 0 \leq \left| \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} - \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}\,dz \right|$$

$$= \left| \frac{1}{2\pi i}\int_C f(z)\cdot\frac{\Delta z}{(z-z_0)^2(z-z_0-\Delta z)}\,dz \right| \leq \frac{M}{2\pi}\cdot L\cdot\frac{|\Delta z|}{d^2\,(d-|\Delta z|)} \to 0$$

as $\Delta z \to 0$.

Thus $f'(z_0) = \dfrac{1}{2\pi i}\displaystyle\int \frac{f(z)}{(z-z_0)^2}\,dz$ $\boxed{\times}$.

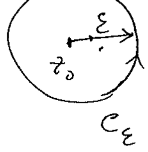
Example

$$\int_{|z|=2} \frac{\sin z}{(z-1)^2}\,dz = 2\pi i\,(\sin z)'\Big|_{z=1} = 2\pi i\cos 1.$$

§52. Some consequences of the extension

Theorem 1. if f is analytic/holomorphic at a point z_0, then $f^{(n)}$ is — ______ $\forall\, n = 1, 2, \dots$

Proof $\exists\, \varepsilon > 0$ s.t. $f \in Hol(\overline{D_\varepsilon(z_0)})$



$$f(z) = \frac{1}{2\pi i} \int_{C_\varepsilon} \frac{f(s)}{s - z}\, ds \quad \forall\, z \in D_\varepsilon(z_0)$$

$$f'(z) = \frac{1}{2\pi i} \int_{C_\varepsilon} \frac{f(s)}{(s - z)^2}\, ds \quad \forall\, z \in D_\varepsilon(z_0)$$

$$\exists\, (f'(z))' = f''(z) = \frac{2!}{2\pi i} \int_{C_\varepsilon} \frac{f(s)}{(s - z)^3}\, ds \quad \forall\, z \in D_\varepsilon(z_0)$$

$$\Rightarrow \quad f' \in Hol(D_\varepsilon(z_0)) \quad ☒$$

- - - -

Similarly $f^{(n)} \in Hol(D_\varepsilon(z_0))$ $\forall\, n$.

Cor. if $f = u(x,y) + i v(x,y)$ is analytic at $z = (x + iy) = (x, y)$ & $\cancel{f}' \in C^0(D_\varepsilon(z))$, then $f, u, v \in C^\infty(D_\varepsilon(z))$ $(\exists\, \varepsilon > 0)$

$\left(\exists\, f^{(n)}(z),\ \exists\, \frac{\partial^{m+n} u}{\partial x^m \partial y^n}(z),\ \exists\, \frac{\partial^{m+n} v}{\partial x^m \partial y^n}(z) \quad \forall\, m, n \in \mathbb{N} \right)$.

Theorem 2 (E. Morera, 1856–1909). Let $f \in C^0(D)$, $D \subset \mathbb{C}$ domain. If $\int_C f(z)\, dz = 0$ $\forall$ closed contour C in D, then $f \in Hol(D)$.

Proof $\int_C f(z)\, dz = 0$ $\forall\, C \subset D$ $\xRightarrow{\text{Sec. 44}}$ $\exists\, F \in Hol(D)$ s.t. $F'(z) = f(z)$ $\forall\, z \in D$ $\xrightarrow{\text{by Thm 1}}$ $\underset{F'}{\underset{\parallel}{f}} \in Hol(D)$ ☒

Theorem 3 (Cauchy's inequality) If $f \in \mathrm{Hol}(\overline{D_R(z_0)})$, then

$$|f^{(n)}(z_0)| \leq \frac{n!}{R^n} M_R, \quad M_R := \max_{z \in C_R} |f(z)|, \quad C_R = \{|z-z_0| = R\}, \; R>0$$

Proof

$$|f^{(n)}(z_0)| = \left| \frac{n!}{2\pi i} \int_{C_R} \frac{f(z)}{(z-z_0)^{n+1}} dz \right|$$

$$\leq \frac{n!\, M_R}{2\pi} \cdot \frac{1}{R^{n+1}} \cdot 2\pi R = \frac{n!\, M_R}{R^n}.$$

Example.

$$\int_{|z-i|=1} \frac{dz}{z^2+1} = ?$$

---

53. Liouville's theorem and the fundamental theorem of algebra

Theorem 1 (Liouville).

If $f \in \mathrm{Hol}(\mathbb{C})$ and is bounded on $\mathbb{C}$, then $f = \mathrm{const}$.

Proof. $\exists M > 0$ s.t. $|f(z)| \leq M \;\forall z \in \mathbb{C}$

$$f'(z) = \frac{1}{2\pi i} \oint_{|s-z|=R} \frac{f(s)}{s-z} dz$$

$$0 \leq |f'(z)| \leq \frac{M}{2\pi} \cdot \frac{1}{R} \cdot 2\pi R = \frac{M}{R}$$

$$R \to +\infty \qquad 0$$

by the Sandwich theorem (Squeeze theorem)

Thus $f'(z) = 0 \;\forall z \in \mathbb{C} \Rightarrow f = \mathrm{const}.$

Theorem 2 (Fundamental theorem of algebra).

Any polynomial $P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0 \quad (a_n \neq 0)$ of degree n $(n \geq 1)$ has at least one zero, i.e., $\exists z_0 \in \mathbb{C}$ s.t. $P(z_0) = 0$

Proof (by contradiction)

Suppose that $P(z) \neq 0 \ \forall z \in \mathbb{C}$.

$$f(z) := \frac{1}{P(z)} \in Hol(\mathbb{C})$$

Claim. f is bounded

Proof . Since $\lim\limits_{z \to \infty} f(z) = 0$, $\exists R > 0$ s.t. $|f(z)| \leq 1 \ \forall |z| \geq R$.

. $M_1 := \max\limits_{|z| \leq R} |f(z)| < +\infty$.

. $M := \max\{M_1, 1\}$. Then

. $|f(z)| \leq M \ \forall z \in \mathbb{C}$ $\square$

$\xrightarrow{\text{by Liouville's thm}}$ $f = const \Rightarrow P = const$.

it is impossible. $\boxtimes$

Remark 1) $\exists z_1$ s.t $P(z_1) = 0 \Rightarrow$

$P(z) = (z - z_1) P_1(z)$, $\exists z_2 \in \mathbb{C}$ s.t

$= (z - z_1)(z - z_2) P_2(z) = \dots$

$\dots$

$= a_n (z - z_1)(z - z_2) \dots (z - z_n)$

Cor. P has n zeros.

2) $P(z) = a_n z^n + \underbrace{\left(\frac{a_{n-1}}{z} + \dots + \frac{a_n}{z^n}\right)}_{g(z)} . z^n$

$= (a_n + g(z)) z^n$

$g(z) \to 0$ as $z \to \infty$ $\Rightarrow \exists R > 0$ s.t. $|g(z)| \le \frac{|a_n|}{2}$ $\forall |z| \ge R$.

$(R > 1)$

Thus $|P(z)| \ge (|a_n| - |g(z)|)|z|^n$

$$\ge \left(|a_n| - \frac{|a_n|}{2}\right) R^n = \frac{|a_n| R^n}{2}$$

$$\Rightarrow |f(z)| = \left|\frac{1}{P(z)}\right| \le \frac{2}{|a_n| R^n} \quad \forall |z| \ge R.$$

On the other hand, f is bounded on $\overline{D_R(0)} = \{ |z| \le R \}$.

, i.e, $\exists M_1 > 0$ s.t $|f(z)| \le M_1$ $\forall |z| \le R$.

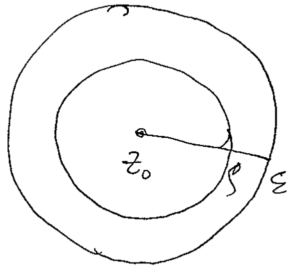
Thus $|f(z)| \le M := \max\left\{ M_1, \frac{2}{|a_n| R^n} \right\}$ $\forall z \in \mathbb{C}$ $\boxtimes$.

54. Maximum modulus principle

Lemma. Suppose that $f \in Hol(D_\varepsilon(z_0))$ $(\varepsilon > 0)$ and $|f(z)| \le |f(z_0)|$ $\forall |z - z_0| < \varepsilon$. Then $f(z) \equiv f(z_0)$ $\forall z \in D_\varepsilon(z_0)$.

Proof

- $0 < \rho < \varepsilon$
- $$f(z_0) = \frac{1}{2\pi i} \int_{|z-z_0|=\rho} \frac{f(z)}{z - z_0}\, dz$$



$$= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \rho e^{i\theta})\, d\theta \quad (1)$$

$$(1) \Rightarrow |f(z_0)| \le \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})|\, d\theta \underset{\text{by assumption}}{\le} \frac{1}{2\pi} \int_0^{2\pi} |f(z_0)|\, d\theta = |f(z_0)|$$

Thus $$|f(z_0)| = \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})|\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} |f(z_0)|\, d\theta$$

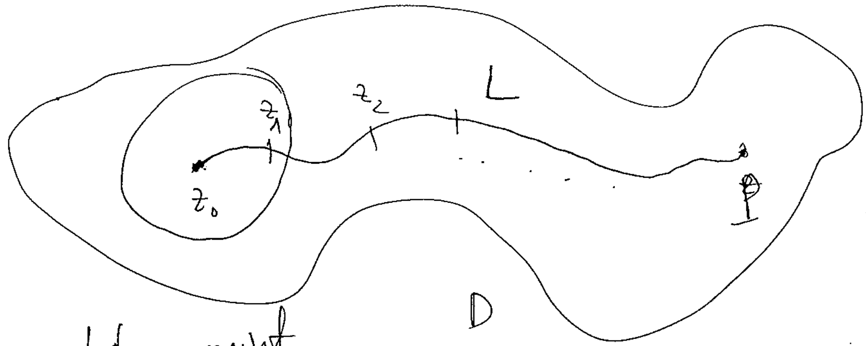
$$\Rightarrow 0 \le \int_0^{2\pi} \underbrace{\left(-|f(z_0 + \rho e^{i\theta})| + |f(z_0)|\right)}_{g(\theta) \ge 0} d\theta = 0 = \int_0^{2\pi} g(\theta)\, d\theta.$$

Since $g(\theta) \ge 0$ $\forall 0 \le \theta \le 2\pi$ & $\int_0^{2\pi} g(\theta)\, d\theta = 0$, $g(\theta) \equiv 0$

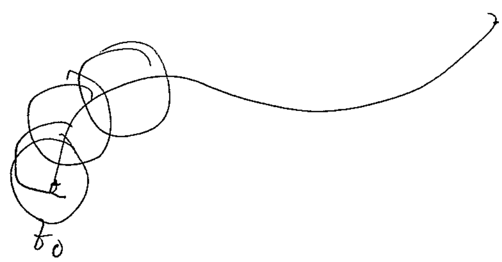
$\Rightarrow |f(z)| = |f(z_0)|$ $\forall |z - z_0| = \rho$ $\forall 0 < \rho < \varepsilon$ $\Rightarrow$ $|f(z)| = $ const $\underset{\text{exercise}}{\Longrightarrow}$ $f(z) = $ const $= f(z_0)$

Theorem (Maximum modulus principle) Let $D \subset \mathbb{C}$ be a domain.
If $f \in Hol(D)$ & $f \not\equiv$ const, then f has no maximum value in D,
i.e., $\nexists z_0 \in D$ s.t. $|f(z)| \le |f(z_0)|$ $\forall z \in D$.

Proof (by contradiction)
Suppose that $\exists z_0 \in D$ s.t. $|f(z)| \le |f(z_0)|$ $\forall z \in D$.



Let $p \in D$ be an arbitrary point.
Let L be a polygonal line, $L \subset D$ (L is a curve)
$d := dist(L, \partial D)$.
Then $\exists \underset{\overset{\|}{z_0}}{z_0}, z_1, z_2, \ldots, z_{n-1}, z_n = p \in L$ such that
$|z_{j+1} - z_j| < d \quad \forall\ 0 \le j \le n-1$.



$N_0 = D_d(z_0), \ldots, N_{n-1} = D_d(z_{n-1})$.

Note that $z_{j+1} \in N_j$.
by Lemma $\Longrightarrow$ $f(z) \equiv f(z_0)$ $\forall z \in N_0 \ni z_1$. Thus
$|f(z)| \le |f(z_1)| = |f(z_0)|$ $\forall z \in N_1$ $\xrightarrow{\text{by lemma}}$ $f(z) \equiv f(z_0)$ $\forall z \in N_1 \ni z_2$

$\ldots$

$f(z) \equiv f(z_0)$ $\forall z \in N_{n-1} \ni p \Rightarrow f(p) = f(z_0)$
Thus $f(z) =$ const $\boxtimes$.

<u>Corollary</u>. Let $D \Subset \mathbb{C}$ be a $\overset{\text{bdd}}{\wedge}$ domain. Suppose that $f \in Hol(D) \cap C^0(\overline{D})$. Then the maximum value of $|f(z)|$ in $\overline{D}$ occurs on ∂D and never in D.

Proof (easy!)

<u>Remark</u>

$$\max_{z \in \overline{D}} |f(z)| = \max_{z \in \partial D} |f(z)| > |f(z)| \; \forall z \in D.$$

<u>Corollary</u>. Let $D \Subset \mathbb{C}$ be a bounded domain. Suppose that $u \in \mathcal{H}(D) \cap C^0(\overline{D})$. Then the maximum value of u in $\overline{D}$ occurs on ∂D and never in D.

Proof $u \in \mathcal{H}(D) \Rightarrow \exists f \in \mathcal{H}(D) \cap C^0(\overline{D})$ such that $f(z) = u(x,y) + i v(x,y)$. Then

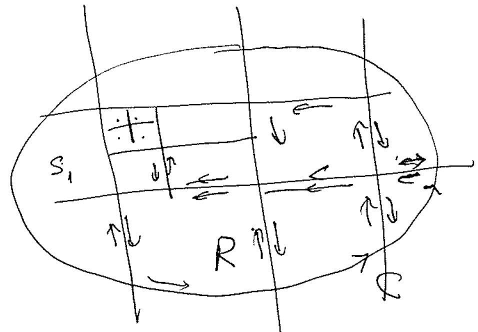
$$g(z) := e^{f(z)}, \quad |g(z)| = e^{\operatorname{Re} f(z)} = e^{u(x,y)}.$$

Since $g \in Hol(D) \cap C^0(\overline{D})$, $\max_{z \in \overline{D}} |g(z)| = \max_{z \in \partial D} |g(z)| > |g(z)| \; \forall z \in D$

$$\Rightarrow \max_{(x,y) \in \overline{D}} u(x,y) = \max_{(x,y) \in \partial D} u(x,y) > u(x,y) \; \forall (x,y) \in D \; \boxtimes.$$

Idea of proof of Cauchy - Goursat's theorem

- f is holomorphic in $\overline{R}$ (in a nbd of R)



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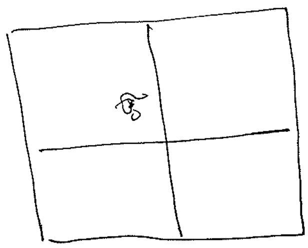
Lemma For every $\varepsilon > 0$, R can be covered with a finite number of squares and partial squares $(q_1, \dots, q_n)$. Then $\exists\, z_j \in q_j$ s.t.

$$\left| \frac{f(z) - f(z_j)}{z - z_j} - f'(z_j) \right| < \varepsilon \quad \forall z \in q_j \qquad (*)$$

— the lemma ~~does~~ is not true.

Suppose $\nexists\, z_{j_0}$ s.t.

$$\left| \frac{f(z) - f(z_{j_0})}{z - z_{j_0}} - f'(z_{j_0}) \right| < \varepsilon \quad \forall z \in q_j$$



$\sigma_0 = q_{j_0}$, subdivide σ_0 ~~in~~ into 4 smaller square

$\Longrightarrow$ $\exists$ subsquare σ_1 s.t. $\nexists\, z_1 \in \sigma_1$ satisfying

$$\left| \frac{f(z) - f(z_1)}{z - z_1} - f'(z_1) \right| < \varepsilon \quad \forall z \in \sigma_1$$

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$\exists\, \sigma_0, \sigma_1, \dots, \sigma_n, \dots$

Note $\operatorname{diam}(\sigma_n) \to 0$ & $\sigma_0 \supset \sigma_1 \supset \dots \supset \sigma_n \supset \dots$

$\xrightarrow{\text{by Cantor's theorem}}$ $\bigcap_{n=0}^{\infty} \sigma_n = \{z_0\}$.

Since $\exists\, f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$, $\exists\, \delta > 0$ s.t.

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon \quad \forall\, |z - z_0| < \delta$$

$\Rightarrow \quad \exists\ \sigma_n \subset D_\delta(z_0)$ but
$$\underset{z_0}{\underset{\Psi}{}}$$

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon \quad \forall\ z \in \sigma_n$$

it is a contradiction $\boxed{X}$

Proof of Theorem

- $\exists$ squares and partial squares $q_1, \dots, q_n$ s.t. $\bigcup_{j=1}^{n} q_j \supset R$

- $\exists\ z_j \in q_j$ s.t.

$$\left| \frac{f(z) - f(z_j)}{z - z_j} - f'(z_0) \right| < \varepsilon \quad \forall\ z \in q_j.$$

- $$\delta_j(z) = \frac{f(z) - f(z_j)}{z - z_j} - f'(z_j) \quad (z \neq z_j)$$

$$\delta_j(z_j) = 0 \qquad |\delta_j(z)| < \varepsilon \quad \forall z \in q_j$$

$$\Rightarrow \quad f(z) = f(z_j) + (z - z_j) f'(z_j) + \delta_j (z - z_j) \qquad \lim_{z \to z_j} \delta_j(z) = f'(z_j) - f'(z_j) = 0$$

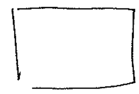
$$f(z) = f(z_j) - z_j f'(z_j) + f'(z_j) \cdot z + (z - z_j)\delta_j(z)$$

$$C_j = \partial q_j$$

$$\int_{C_j} f(z)dz = f(z_j) - z_j f'(z_j) \underbrace{\int_{C_j} dz}_{= 0} + f'(z_j) \underbrace{\int_{C_j} z dz}_{= 0} + \int_{C_j} (z - z_j)\delta_j(z)$$

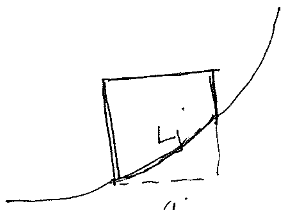
$$= \int_{C_j} (z - z_j)\delta_j(z)$$

$$\int_C f(z)\,dz = \sum_{j=1}^{n} \int_{C_j} f(z)\,dz = \sum_{j=1}^{n} \int_{C_j} (z-z_j)\,\delta_j(z)\,dz$$



q_j

$$\left|\int_C f(z)\,dz\right| \le \sum_{j=1}^{n} \int_{C_j} \underbrace{|z-z_j|\,\delta_j(z)}_{\le \sqrt{2}\, s_j\, \varepsilon}\,dz$$



q_j

$$\le \sum_{j=1}^{n} \sqrt{2}\, s_j\, \varepsilon \cdot (4 s_j + L_j)$$

$$\le \sum_{j=1}^{n} 4\sqrt{2}\, A_j \varepsilon + \sqrt{2}\, s_j \cdot L_j \cdot \varepsilon$$

$L_j = |q_j \cap C|$

$A_j = s_j^2$

$$0 \le \left|\int_C f(z)\,dz\right| \le \left(4\sqrt{2}\, S^2 + \sqrt{2}\, S L\right)\varepsilon \to 0$$

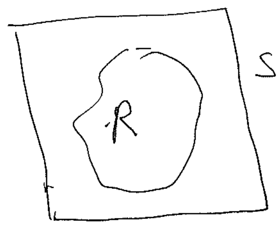
$\sum A_j$

$L_j \le L = \text{length}(C)$

0

$\sum A_j \le S^2$

$$\Rightarrow \quad \int_C f(z)\,dz = 0 \qquad \boxed{\times}$$



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Chapter 5. Series

$$f \in Hol(D) \rightsquigarrow f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \cdots + \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n + \cdots$$

55. ## Convergence of sequences

$\{z_n\}_{n=1}^{\infty} \subset \mathbb{C}$ $\left(\begin{array}{l} f : \mathbb{N}^* \to \mathbb{C} \text{ is a map} \\ z_n := f(n) \ \forall\, n = 1, 2 \ldots \end{array} \right)$

Def. $\lim_{n\to\infty} z_n = z \overset{def}{\Longleftrightarrow} \forall \varepsilon > 0 \ \exists\, n_0 \in \mathbb{N}$ such that $|z_n - z| < \varepsilon$ for any $n \geq n_0$.

- z : the limit of $\{z_n\}$
- $\{z_n\}$ is convergent if $\exists \lim z_n$
- $\{z_n\}$ is divergent if $\nexists \lim z_n$.

Note. A limit z is unique if it exists (Exercise)

Proof Suppose that $z_n \longrightarrow z \in \mathbb{C}$ as $n \to \infty$.
$z_n \longrightarrow z' \in \mathbb{C}$, $z \neq z'$

$\forall \varepsilon > 0 \ \exists\, n_1$ s.t. $|z_n - z| < \varepsilon/2 \quad \forall\, n \geq n_1$

$\exists\, n_2$ s.t. $|z_n - z'| < \varepsilon/2 \quad \forall\, n \geq n_2$

$\varepsilon_0 := |z - z'|$

$n_0 := \max\{n_1, n_2\}$, choose $n > n_0$. Then

$$0 < \varepsilon_0 = |z - z'| \leq |z_n - z| + |z_n - z'| < \frac{\varepsilon_0}{2} + \frac{\varepsilon_0}{2} = \varepsilon_0$$

This is a contradiction $\boxtimes$

Theorem. $\lim_{n\to\infty} z_n = z \iff \begin{cases} \lim_{n\to\infty} x_n = x \\ \lim_{n\to\infty} y_n = y \end{cases}$

, $z_n = x_n + i y_n$, $z = x + iy$.

Proof. $z_n - z = (x_n - x) + i(y_n - y)$

Use:

$\max\{|x_n - x|, |y_n - y|\} \leq |z_n - z| \leq |x_n - x| + |y_n - y|$

"$\Rightarrow$" Suppose that $\lim z_n = z$. Then $\forall \varepsilon > 0$ $\exists n_0$ s.t.

$|z_n - z| < \varepsilon$ $\forall n \geq n_0$. Then $|x_n - x| < \varepsilon$ & $|y_n - y| < \varepsilon$ $\forall n \geq n_0$.

This implies that $\lim x_n = x$ & $\lim y_n = y$.

"$\Leftarrow$" Suppose that $\lim x_n = x$ & $\lim y_n = y$.

$\forall \varepsilon > 0$ $\exists n_1$ s.t $|x_n - x| < \varepsilon/2$ ~~& $|y_n$~~ $\forall n \geq n_1$

$\exists n_2$ s.t $|y_n - y| < \varepsilon/2$ $\forall n \geq n_2$.

$n_0 := \max\{n_1, n_2\}$. Then

$$|z_n - z| \leq |x_n - x| + |y_n - y| < \frac{\varepsilon}{2} + \varepsilon/2 = \varepsilon \quad \forall n \geq n_0 .$$

This implies that $\lim_{n\to\infty} z_n = z := x + iy$.

Example
$$\lim_{n\to\infty} \frac{n+1}{n-1} + i\left(1 + \frac{1}{n}\right)^n = \lim \frac{n+1}{n-1} + i \lim \left(1 + \frac{1}{n}\right)^n = 1 + i \cdot e .$$

56. Convergence of series

Let $\{z_n\} \subset \mathbb{C}$ be a sequence.

$\sum_{n=1}^{\infty} z_n$: an infinite series

$$\left.\begin{aligned} S_1 &= z_1 \\ S_2 &= z_1 + z_2 \cdots \\ &\cdots \\ S_n &= z_1 + \cdots + z_n \end{aligned}\right\} \text{ partial sums}, \quad \{S_n\} \subset \mathbb{C}$$

Def $\sum_{n=1}^{\infty} z_n = S$ (converges to S) if $\lim_{n\to\infty} S_n = S$.

<u>Theorem</u> $z_n = x_n + iy_n$, $S = X + iY$.

Then $\sum_{n=1}^{\infty} z_n = S \iff \sum_{n=1}^{\infty} x_n = X$ & $\sum_{n=1}^{\infty} y_n = Y$.

Proof $S_n = z_1 + \cdots + z_n = X_n + iY_n$

$$S_n \to S \iff X_n \to X \ \& \ Y_n \to Y. \boxed{\times}$$

<u>Corollary 1.</u> $\sum_{n=1}^{\infty} z_n$ is convergent $\Rightarrow$ $z_n \to 0$ as $n \to \infty$.

Proof. $\sum z_n$ is convergent $\Rightarrow$ $\sum x_n$ & $\sum y_n$ are convergent

$\Rightarrow x_n \to 0$ & $y_n \to 0 \Rightarrow z_n \to 0$.

<u>Cor. 2.</u> $\sum_{n=1}^{\infty} |z_n|$ is convergent $\Rightarrow$ $\sum z_n$ is convergent.

!!

$\sum z_n$ is absolutely convergent

Proof $\sum |z_n|$ is —— $\iff \sum_{n=1}^{\infty} \sqrt{x_n^2 + y_n^2} < +\infty$

$\Rightarrow$ $\sum_{n=1}^{\infty} |x_n| < \sum_{n=1}^{\infty} \sqrt{x_n^2 + y_n^2} < +\infty \Rightarrow \sum x_n$ converges

$\sum_{n=1}^{\infty} |y_n| < \sum_{n=1}^{\infty} \sqrt{x_n^2 + y_n^2} < +\infty \Rightarrow \sum y_n$ ——

Thus $\sum z_n$ is convergent.

Ex 1) $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ if $|z|<1$

$$S_n = 1+z+\cdots+z^{n-1} = \frac{1-z^n}{1-z} \longrightarrow \frac{1}{1-z} \text{ as } n\to\infty$$

$\forall\ |z|<1$.

2) a) $\sum_{n=1}^{\infty} r^n \cos n\theta = \frac{r\cos\theta - r^2}{1-2r\cos\theta+r^2}$

& b) $\sum_{n=1}^{\infty} r^n \sin n\theta = \frac{r\sin\theta}{1-2r\cos\theta+r^2}$

$|r|<1$
$\forall\ \theta\in\mathbb{R}$.

a.) $\operatorname{Re}\sum_{n=0}^{\infty} z^n = \operatorname{Re}\frac{1-\bar{z}}{|1-z|^2}$, $\quad z = re^{i\theta}$

$|1-z|^2 = (1-r\cos\theta)^2 + r^2\sin^2\theta$
$= 1-2r\cos\theta+r^2$.

$$\Rightarrow \sum_{n=0}^{\infty} r^n\cos n\theta = \frac{1-r\cos\theta}{1-2r\cos\theta+r^2}$$

b) Similarly,

57, 58, 59. Taylor series

Theorem (Taylor). If $f \in Hol(D_{R_0}(z_0))$, then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n, \quad |z-z_0| < R_0,$$

where $a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \cdot \oint_{|s-z_0|=\rho} \frac{f(s)}{(s-z_0)^{n+1}} ds, \; 0<\rho<R_0$

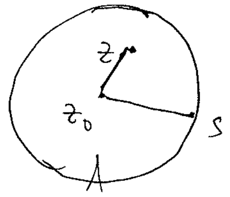
~~f can~~

- $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n$: Taylor series
- $z_0 = 0$, $f(z) = \sum_{n=0}^{\infty} \frac{f^{n}(0)}{n!} z^n$: Maclaurin series.

Proof. $0 < \rho < R_0$.

- $\underset{|z-z_0|<\rho}{f(z)} = \frac{1}{2\pi i} \oint_{C_\rho} \frac{f(s)}{s-z} ds, \quad C_\rho = \{ |s-z_0| = \rho \}$

(Cauchy integral formula)



- $\frac{1}{s-z} = \frac{1}{(s-z_0)-(z-z_0)} = \frac{1}{s-z_0} \cdot \frac{1}{1 - \frac{z-z_0}{s-z_0}}$

$$= \frac{1}{s-z_0} \sum_{n=0}^{\infty} \left(\frac{z-z_0}{s-z_0} \right)^n, \quad \left| \frac{z-z_0}{s-z_0} \right| < 1$$

$$= \sum_{n=0}^{N-1} \frac{(z-z_0)^n}{(s-z_0)^{n+1}} + \frac{(z-z_0)^N}{(s-z_0)^{N+1}} \cdot \frac{1}{1 - \frac{z-z_0}{s-z_0}}$$

$$\frac{1}{2\pi i} \int_{C_\rho} \frac{f(s)}{s-z} ds = \sum_{n=0}^{N-1} \left(\frac{1}{2\pi i} \cdot \oint_{C_\rho} \frac{f(s)}{(s-z_0)^{n+1}} ds \right) (z-z_0)^n + R_N(z)$$

$$= \sum_{n=0}^{N-1} a_n (z-z_0)^n + R_N(z).$$

$$0 \leq \left| R_N(z) \right| = \left| \frac{1}{2\pi i} \oint_{C_\rho} \frac{f(s)}{1 - \frac{z-z_0}{s-z_0}} \cdot \frac{(z-z_0)^N}{(s-z_0)^{N+1}} \, ds \right|$$

$$0 \leq \left| R_N(z) \right| \leq \frac{1}{2\pi} \cdot M_\rho \cdot \left| \frac{z-z_0}{\rho} \right|^N \cdot \frac{1}{\rho - |z-z_0|}$$

$N \to +\infty$

0

$$\boxed{|f(z)| \leq M_\rho \quad \forall z \in C_\rho}$$

$$\Rightarrow \lim_{N\to\infty} R_N(z) = 0$$

$$\text{So,} \quad f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n, \qquad |z-z_0| < R_0.$$

<u>Example</u>.

1) $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, $z \in \mathbb{C}$, $f(z) = e^z$, $f^{(n)}(0) = 1$.

2) $\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$ $z \in \mathbb{C}$, $f = \cos z$, $f^{(n)}(0) = \begin{cases} 0 & \text{if } n = 2k+1 \\ (-1)^k & \text{if } n = 2k. \end{cases}$

3) $\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$

4) $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, $|z| < 1$

5) $\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n$, $|z| < 1$

6) $\sinh z = -i \sin(iz) = -i \sum_{n=0}^{\infty} (-1)^n \cdot \frac{(iz)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$ $\forall z \in \mathbb{C}$.

7) $\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \left(= \cos(iz) \right)$ $\forall z$.

8) $\mathrm{Log}(1+z) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n}$, $|z| < 1$.

Example Expand $f(z) = \dfrac{1}{z^2+3z+2}$ into Taylor series at $z=3$

$$f(z) = \frac{1}{z^2+3z+2} = \frac{1}{(z+1)(z+2)} = \frac{1}{z+1} - \frac{1}{z+2}$$

$$= \frac{1}{4+(z-3)} - \frac{1}{5+(z-3)}$$

$$= \frac{1}{4\left(1+\frac{z-3}{4}\right)} - \frac{1}{5}\cdot\frac{1}{1+\frac{z-3}{5}}$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{(z-3)^n}{4^{n+1}} - \sum (-1)^n\frac{(z-3)^n}{5^{n+1}}$$

$$= \sum_{n=0}^{\infty}(-1)^n\cdot\left(\frac{1}{4^{n+1}} - \frac{1}{5^{n+1}}\right)\cdot(z-3)^n \quad \forall, \quad |z-3| < 4.$$

60, 61, 62

Laurent series

an annular domain / annulus

Theorem (Laurent). $V := \{ z \in \mathbb{C} : R_1 < |z - z_0| < R_2 \}$, $0 \le R_1 < R_2 \le +\infty$. Suppose that $f \in Hol(V)$. Then $\forall z \in V$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}, \quad R_1 < |z-z_0| < R_2,$$

where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{1}{2\pi i} \int_{|z-z_0|=\rho} \frac{f(z)}{(z-z_0)^{n+1}} dz, \quad R_1 < \rho < R_2, \quad n = 0, 1, 2 \ldots$$

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{-n+1}} dz = \frac{1}{2\pi i} \int_{|z-z_0|=\rho} \frac{f(z)}{(z-z_0)^{-n+1}} dz, \quad n = 1, 2, 3 \ldots$$

C : any positively oriented simple closed contour around z_0 in V.

Remark. $f(z) = \sum_{n=-\infty}^{+\infty} c_n (z-z_0)^n$,

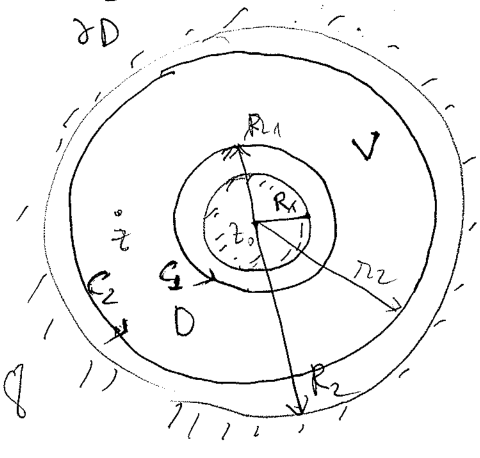
$$c_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz \quad \forall n \in \mathbb{Z}$$

Proof. Let $R_1 < r_1 < r_2 < R_2$ $\quad (r_1 \to R_1^+, r_2 \to R_2^-)$

$D := \{ r_1 < |z-z_0| < r_2 \}$, $\partial D = C_2 - C_1$

$f \in Hol(\overline{D}) \Rightarrow$

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(s)}{s-z} ds = \underbrace{\frac{1}{2\pi i} \oint_{|s-z_0|=r_2} \frac{f(s)}{s-z} ds}_{=A(z)} - \underbrace{\frac{1}{2\pi i} \oint_{|s-z_0|=r_1} \frac{f(s)}{s-z} ds}_{B(z)}$$



- $$A(z) = \frac{1}{2\pi i} \oint_{|s-z_0|=r_2} \frac{f(s)}{s-z}\, ds \qquad r_1 < |z-z_0| < r_2$$

$$\frac{1}{s-z} = \frac{1}{s-z_0-(z-z_0)} = \frac{1}{s-z_0} \cdot \frac{1}{1-\frac{z-z_0}{s-z_0}}$$

$$= \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(s-z_0)^{n+1}} \qquad \text{Sec. 57} \sim 59.$$

$$A(z) = \frac{1}{2\pi i} \oint_{|s-z_0|=r_2} f(s) \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(s-z_0)^{n+1}}\, ds \overset{\downarrow}{=\!=} \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

- $$B(z) = \frac{1}{2\pi i} \int_{|s-z_0|=r_1} \frac{f(s)}{s-z}\, ds$$

$$\frac{1}{s-z} = \frac{1}{s-z_0-(z-z_0)} = \frac{-1}{z-z_0} \cdot \frac{1}{1-\frac{s-z_0}{z-z_0}}$$

$$= -\frac{1}{z-z_0} \cdot \sum_{n=0}^{\infty} \left(\frac{s-z_0}{z-z_0}\right)^n, \text{ since } \left|\frac{s-z_0}{z-z_0}\right| < 1$$

$$= -\sum_{n=0}^{N-1} \frac{(s-z_0)^n}{(z-z_0)^{n+1}} - \left(\frac{s-z_0}{z-z_0}\right)^N \cdot \frac{1}{z-z_0-(s-z_0)}.$$

$$B(z) = \frac{1}{2\pi i} \oint_{|s-z_0|=r_1} \frac{f(s)}{s-z}\, ds = -\sum_{m=1}^{N} \frac{b_m}{(z-z_0)^m} - R_N(z)$$

$$0 \le |R_N(z)| = \left| \frac{1}{2\pi i} \int_{(s-z_0)=r_1} \left(\frac{s-z_0}{z-z_0}\right)^N \cdot \frac{f(s)}{z-z_0-(s-z_0)}\, ds \right|$$

$$0 \le |R_N(z)| \le \frac{1}{2\pi} \cdot \left(\frac{r_1}{|z-z_0|}\right)^N \cdot M_{r_1} \cdot \frac{1}{|z-z_0|-r_1}$$

$N \to +\infty$ $\Rightarrow$ $R_N(z) \Longrightarrow 0$ as $N \to +\infty$.

0

$$\Rightarrow f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \quad \forall\, r_1 < |z-z_0| < r_2$$

$$\hookrightarrow f(z) = \sum_{n=0}^{\infty} \qquad \forall\, R_1 < |z-z_0| < R_2.$$

$\boxtimes$

Examples.

1) $$e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{n!\, z^n}, \quad 0 < |z| < +\infty$$

2) $$f(z) = \frac{1}{z^2+3z+2},$$ Find the Laurent series of f in the domains

a) $V_1 = \{1 < |z| < 2\}$

b) $V_2 = \{+\infty > |z| > 2\}$

c) $V_3 = \{0 < |z| < 1\}$.

a) $$f(z) = \frac{1}{z+1} - \frac{1}{z+2}$$

$$= \frac{1}{\cancel{z}\cdot z} \frac{1}{1+\frac{1}{z}} - \frac{1}{2}\cdot\frac{1}{1+\frac{z}{2}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{2^{n+1}}, \quad 1 < |z| < 2$$

$|z| > 2$

b) $$f(z) = \frac{1}{z+1} - \frac{1}{z+2}$$

$$= \frac{1}{z}\cdot\frac{1}{1+\frac{1}{z}} - \frac{1}{z}\cdot\frac{1}{1+\frac{2}{z}}$$

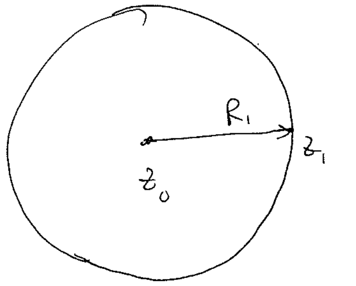
$$= \sum_{n=0}^{\infty} (-1)^n \left(1 - 2^n\right)\cdot\frac{1}{z^{n+1}}$$

c) $$f(z) = \sum_{n=0}^{\infty} \left(1 - \frac{1}{2^{n+1}}\right)(-1)^n z^n, \quad 0 < |z| < 1$$

63. Absolute and uniform convergence of power series

- Recall. $\sum_{n=1}^{\infty} f_n(z)$ is ~~uniformly~~ absolutely convergent if $\sum_{n=1}^{\infty} |f_n(z)| < +\infty$

a) Theorem 1. If $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges at $z = z_1 \neq z_0$.
then it is absolutely convergent at each point $z \in \{|z-z_0| < R_1\}$
where $R_1 = |z_1 - z_0| > 0$.



Proof.

- Since $\sum_{n=1}^{\infty} a_n(z_1-z_0)^n$ is convergent,
$a_n(z_1-z_0)^n \longrightarrow 0$ as $n \to \infty \Rightarrow \exists M > 0$ s.t. $|a_n(z_1-z_0)^n| \leq M$ $\forall n = 0, 1, 2, \dots$

- $$|a_n(z-z_0)^n| = |a_n(z_1-z_0)^n| \cdot \left|\frac{z-z_0}{z_1-z_0}\right|^n \leq M \cdot \rho^n \quad \forall z \in D_{R_1}$$

- $\sum_{n=0}^{\infty} M\rho^n < +\infty$ (since $\rho = \frac{|z-z_0|}{|z_1-z_0|} = \frac{|z-z_0|}{R_1} < 1$)

$\xrightarrow{\text{by M-test}} \sum_{n=0}^{\infty} |a_n(z-z_0)^n| < +\infty \ \forall |z-z_0| < R_1$. $\boxtimes$

b). Radius of convergence

$$R := \sup\left\{ |z-z_0| \;\middle|\; \sum_{n=0}^{\infty} a_n(z-z_0)^n \text{ is convergent} \right\}$$

- R is called the radius of convergence.

$$R \overset{\text{(Exercise)}}{=\!=} \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}} \in [0, +\infty] \quad \text{(Cauchy-Hadamard's formula)}$$

- $\{|z-z_0| = R\}$: the circle of convergence

Def. $\sum_{n=0}^{\infty} f_n(z)$ is uniformly convergent in $D \subset \mathbb{C}$

$\Longleftrightarrow$ $\{S_n(z)\}$ ——— $D \subset \mathbb{C}$

$\Longleftrightarrow$ $\lim_{n\to\infty} \sup_{z\in D} |S_n(z) - S(z)| = 0$.

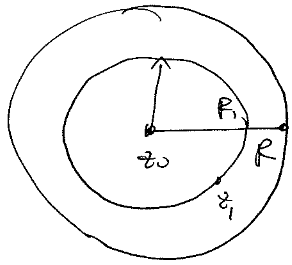
$\Longleftrightarrow$ $\forall \varepsilon > 0 \ \exists\, n_0 = n_0(\varepsilon)$ s.t. $|S_n(z) - S(z)| < \varepsilon$ for $\forall z \in D$, $\forall n \geq n_0$.

$\Longleftrightarrow$ $\forall \varepsilon > 0 \ \exists\, n_0 = n_0(\varepsilon)$ s.t. $|S_{n+k}(z) - S_n(z)| < \varepsilon$ $\forall z \in D$ $\forall n > n_0$ $\forall k \in \mathbb{N}$.

Notation: $\sum_{n=0}^{\infty} f_n(z) \overset{D}{\rightrightarrows} S(z) \Longleftrightarrow S_n(z) \overset{D}{\rightrightarrows} S(z)$.

Theorem 2. Suppose the radius of convergence R of $\sum a_n(z-z_0)^n$ is positive ($R>0$). Then

(1) $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ must be uniformly convergent in $\overline{D_{R_1}}(z_0) = \{|z-z_0| \leq R_1\}$ for all $0 < R_1 < R$.



(2)

Proof.

. Choose $z_1 \in D_R(z_0)$ s.t. $R_1 = |z_1 - z_0|$

. $\sum_{n=0}^{\infty} |a_n(z_1-z_0)^n| < +\infty$ (by Thm 1)

$\underset{\text{by Cauchy's criterion}}{\longrightarrow}$ $\forall \varepsilon > 0 \ \exists\, n_0(\varepsilon)$ s.t $\sum_{m=n}^{n+k} |a_m(z_1-z_0)^m| < \varepsilon$ $\forall n > n_0$ $\forall k \in \mathbb{N}$

. $\left| \sum_{m=n}^{n+k} a_m(z-z_0)^m \right| \leq \sum_{m=n}^{n+k} |a_m(z_1-z_0)^m| < \varepsilon$

$\forall z \in \overline{D_{R_1}}(z_0)$ $\forall n > n_0(\varepsilon)$, $\forall k$

$\Longrightarrow$ $\sum_{n=0}^{\infty} a_n(z-z_0)^n \xrightarrow[\longrightarrow]{\overline{D_{R_1}}(z_0)} S(z)$.

Example. $\displaystyle\sum_{n=0}^{\infty} \left(1+\frac{1}{n}\right)^{n^2} (z+1)^n$

$$\Rightarrow \quad R = \frac{1}{\overline{\lim} \left(1+\frac{1}{n}\right)^n} = \frac{1}{e}.$$

64. Continuity of sums of power series

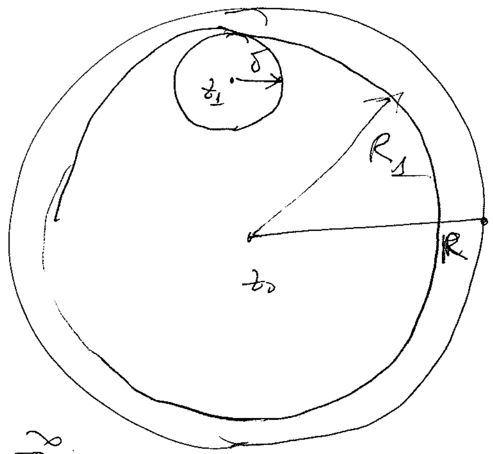
Theorem. R = the radius of convergence of $\sum a_n (z-z_0)^n > 0$

Then $$S(z) := \sum_{n=0}^{\infty} a_n (z-z_0)^n \in C^0(D_R(z_0)).$$

Let $z_1 \in D_R(z_0)$. Fix z_1.

& let $R_1 \in (|z_1 - z_0|, R)$

$|z_1 - z_0| < R_1 < R$.



$$S(z) - S(z_1) = \sum_{n=0}^{\infty} a_n (z-z_0)^n - \sum_{n=0}^{\infty} a_n (z_1 - z_0)^n$$

$$= \sum_{n=0}^{N-1} \left[a_n (z-z_0)^n - a_n (z_1 - z_0)^n\right] + \sum_{n=N}^{+\infty} a_n (z-z_0)^n$$

$$- \sum_{n=N}^{+\infty} a_n (z_1 - z_0)^n$$

- Since $\displaystyle\sum_{n=0}^{\infty} a_n (z-z_0)^n \xrightarrow[\quad]{\overline{D_{R_1}(z_0)}} S(z)$, $\forall \varepsilon > 0 \ \exists N$ s.t.

$$\left| \sum_{n=N}^{+\infty} a_n (z-z_0)^n \right| < \varepsilon/3 \quad \& \quad \left| \sum_{n=N}^{+\infty} a_n (z_1 - z_0)^n \right| < \varepsilon/3 \quad (1)$$
$\forall |z - z_0| \leq R_1$.

- Since $g(z) := \displaystyle\sum_{n=0}^{N} a_n (z-z_0)^n$ is continuous at z_1, $\exists$ $0 < \delta < R_1 - |z_1 - z_0|$ s.t. $|g(z) - g(z_1)| < \varepsilon/3 \quad \forall |z - z_1| < \delta$. (2)

By (1) & (2), $\left|S(z) - S(z_1)\right| \leq \left|g(z) - g(z_1)\right| + \left|\sum_{n=N}^{+\infty} a_n (z - z_0)^n\right| + \left|\sum_{n=N}^{\infty} a_n (z_1 - z_0)^n\right|$

$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \quad \forall \ |z - z_1| < \delta$$

Thus $\lim_{z \to z_1} S(z) = S(z_1)$ & $S \in C^0(D_R(z_0))$.

Remark 1) $f(z) := \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$

$t := \frac{1}{z - z_0} \Rightarrow f(z) = f\left(\frac{1}{t}\right) = \sum_{n=1}^{\infty} b_n t^n,$

$\sum_{n=1}^{\infty} b_n t^n \Longrightarrow f\left(\frac{1}{t}\right)$ on $|t| < R$: the radius of convergence.

$\Rightarrow \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$ is convergent on $\frac{1}{|z - z_0|} < R \Leftrightarrow |z - z_0| > \underset{\overset{\shortparallel}{r}}{\frac{1}{R}}$

2) $f(z) := \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$ is
(Laurent series)

convergent on $\{ r < |z - z_0| < R \} = V$.

& $f \in C^0(V)$.

65. Integration and Differentiation of Power series

$$S(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n, \quad R = \text{the radius of convergence}, \; R > 0$$

Theorem 1. Let C be a contour, $C \subset D_R(z_0) = \{|z-z_0|<R\}$ $(R>0)$

$g: C \to \mathbb{C}$ is continuous.

Then

$$\int_C g(z) \underset{\substack{\uparrow \\ \text{is well defined}}}{S(z)}\, dz = \sum_{n=0}^{\infty} a_n \int_C g(z)(z-z_0)^n \, dz \quad (1)$$

Remark 1) $g(z) \equiv 1 \Rightarrow C =$ a line connecting from z_0 to z,

we get

$$\int_{z_0}^{z} S(s)\, ds = \sum_{n=0}^{\infty} a_n \int_{z_0}^{z} (s-z_0)^n \, ds$$

$$= \sum_{n=0}^{\infty} a_n \frac{(z-z_0)^{n+1}}{n+1}.$$

~~Proof.~~ 2) $g(z) \equiv 1$, C is closed contour. Then

$$(1) \Leftrightarrow \int_C S(z)\, dz = \sum_{n=0}^{\infty} a_n \underbrace{\int_C (z-z_0)^n \, dz}_{=0} = 0$$

$\forall$ closed contour $C \subset D_R(z_0)$.

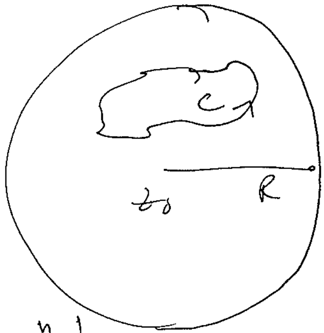
by Morera's theorem $\Longrightarrow$ $S(z) \in Hol(D_R(z_0))$.

Proof: $g(z)S(z) = g(z)\sum_{n=0}^{N-1} a_n(z-z_0)^n + g(z)\sum_{n=N}^{\infty} a_n(z-z_0)^n$

. Let C be a contour in $D_R(z_0)$

L = the length of C

$M := \max_{z\in C} |g(z)|$



. Since

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n \overset{\text{on } C}{\rightrightarrows} S(z)$$

, $\forall \varepsilon > 0$, $\exists N_\varepsilon$ s.t. $\left|\sum_{n=N}^{\infty} a_n(z-z_0)^n\right| < \frac{\varepsilon}{ML}$.

. Then

$$\left|\int_C g(z)S(z)\,dz - \sum_{n=0}^{N-1} a_n\left(\int_C g(z)(z-z_0)^n dz\right)\right| \le \left|\int_C g(z)\sum_{n=N}^{+\infty} a_n(z-z_0)^n\,dz\right|$$

$$\le M.\frac{\varepsilon}{ML}.L = \varepsilon.$$

Thus $\lim_{N\to+\infty} \sum_{n=0}^{N-1} a_n \left(\int_C g(z)(z-z_0)^n dz\right) = \int_C g(z)S(z)\,dz$

$$\| $$

$$\sum_{n=0}^{\infty} a_n\left(\int_C g(z)(z-z_0)^n dz\right) = \int_C g(z)\,S(z)\,dz.$$

—

Corollary. $S(z) \in Hol(D_R(z_0))$.

Example 1) $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \Rightarrow -\mathrm{Log}(1-z) = \sum_{n=0}^{\infty} \frac{z^{n+1}}{n+1}$

$\frac{1}{1-z)}$ $= \sum_{n=1}^{\infty} \frac{z^n}{n}$

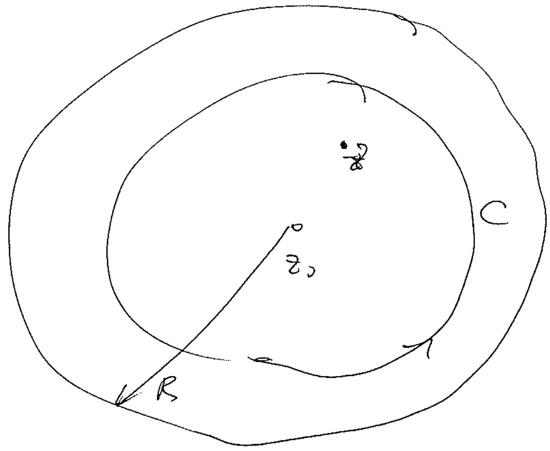
& $\mathrm{Log}(1+z) = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{z^n}{n}$, $|z|<1$.

Theorem 2. $S(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n, \quad |z-z_0| < R, \ R>0.$

Then $S'(z) = \sum_{n=1}^{\infty} a_n n (z-z_0)^{n-1} = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}, \quad \forall z \in D_R(z_0).$

Proof. C is a ~~contour~~ positively-oriented simple closed contour ~~in~~ in $D_R(z_0)$ s.t. z is inside of C.

$$S'(z) = \frac{1}{2\pi i} \int_C \frac{S(s)}{(s-z)^2} ds$$



$$= \int_C \frac{1}{2\pi i (s-z)^2} \cdot S(s)\, ds$$

$$\overset{\text{Thm 1}}{=} \sum_{n=0}^{\infty} a_n \int_C \frac{1}{2\pi i} \cdot \frac{(s-z_0)^n}{(s-z)^2} ds$$

$$= \sum_{n=0}^{\infty} a_n \cdot \left[(s-z_0)^n \right]' \Big|_{s=z} = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1} \boxtimes$$

Example

1) $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \Rightarrow \frac{1}{(1-z)^2} = \sum_{n=1}^{\infty} n (z)^{n-1}$

$\frac{1}{(1+z)^2} = \sum_{n=1}^{\infty} n (-1)^{n-1} z^{n-1}, \quad |z| < 1$

2) $\frac{1}{z^2} = \frac{1}{(1+(z-1))^2} = \sum_{n=1}^{\infty} n (-1)^{n-1} (z-1)^{n-1}, \quad |z-1| < 1$

6.6 Uniqueness of series representations

Theorem 1. $\sum_{n=0}^{\infty} a_n (z-z_0)^n \xrightarrow{D_R(z_0)} f(z)$.

Then the Taylor series expansion of f is $\sum_{n=0}^{\infty} a_n (z-z_0)^n$.

Proof. $f(z) := \sum_{n=0}^{\infty} a_n (z-z_0)^n \in Hol(D_R(z_0))$

$\Rightarrow f(z_0) = a_0$

- $f'(z) = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}$

$\Rightarrow f'(z_0) = 1 a_1 \Rightarrow a_1 = \frac{f'(z_0)}{1}$

...

- $f^{(n)}(z) = \sum_{m=n}^{\infty} m(m-1)\cdots(m-n+1)\, a_m \cdot z^{m-n}$

$\Rightarrow f^{(n)}(z_0) = n!\, a_n \Rightarrow a_n = \frac{f^{(n)}(z_0)}{n!}$ ..

Thus $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ is the Taylor series expansion of f. $\boxtimes$

Another proof.

$f(z) = \sum_{m=0}^{\infty} a_m (z-z_0)^m$

$$f^{(n)}(z_0) = \int_C \frac{n!}{2\pi i} \frac{f(z)}{(z-z_0)^{n+1}}\, dz = \sum_{m=0}^{\infty} a_m \frac{n!}{2\pi i} \int_C \underbrace{\frac{(z-z_0)^m}{(z-z_0)^{n+1}}\, dz}_{= \begin{cases} 2\pi i & \text{if } m=n \\ 0 & \text{if } m \neq n \end{cases}}$$

$= n!\, a_n$. $\boxtimes$

Theorem 2. If $\sum_{n=-\infty}^{+\infty} c_n (z-z_0)^n \xrightarrow{\text{in } V = \{z \mid \cdot - z_0| < R\}} f(z)$

then it is the Laurent series expansion for f in powers of $(z-z_0)^n$ for V. (it is the Laurent series of f about z_0)

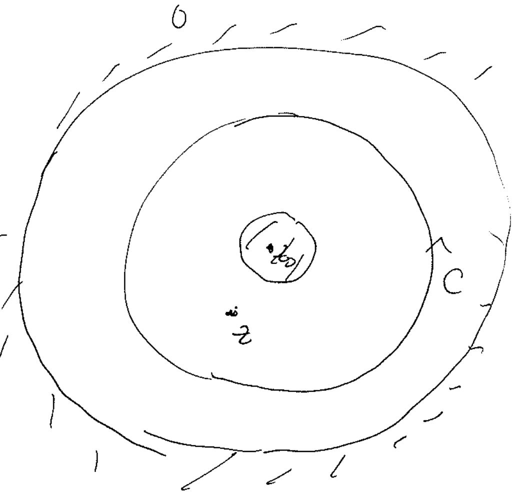
Proof $\quad f(z) := \sum_{m=-\infty}^{\infty} c_m (z-z_0)^m$

$m \in \mathbb{Z}$, $z \in V$, C is a simple closed contour in V

$$\int_C \frac{n!}{2\pi i} \frac{f(z)}{(z-z_0)^{n+1}} dz \overset{\text{thm 1}}{=} \sum_{m=-\infty}^{+\infty} c_m \cdot \frac{n!}{2\pi i} \underbrace{\int_C \frac{(z-z_0)^m}{(z-z_0)^{n+1}} dz}_{0 \text{ if } m \neq n}$$

$$\frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz = n!\, c_n .$$

$$\Rightarrow \quad c_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz \quad \boxtimes$$



6.7. Multiplication and division of power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n, \quad g(z) = \sum_{n=0}^{\infty} b_n (z-z_0)^n, \quad |z-z_0| < R$$

$$f(z) \cdot g(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n$$

$$c_n = \frac{(f \cdot g)^{(n)}(z_0)}{n!} = \sum_{k=0}^{n} \binom{n}{k} \frac{f^{(k)}(z_0) \cdot g^{(n-k)}(z_0)}{n!}.$$

$$= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \cdot \frac{f^{(k)}(z_0) \cdot g^{(n-k)}(z_0)}{n!} = \sum_{k=0}^{n} \frac{f^{(k)}(z_0)}{k!} \cdot \frac{g^{(n-k)}(z_0)}{(n-k)!}$$

$$= \left(\sum_{k=0}^{n} a_k b_{n-k}\right).$$

Example

$$\frac{e^z}{1-z} = \left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots\right)\left(1+z+z^2+\cdots\right)$$

$$= 1 + 2z + \frac{5}{2}z^2 + \cdots \qquad (|z|<1)$$

—

$$\frac{f(z)}{g(z)} = \sum_{n=0}^{\infty} d_n (z-z_0)^n \quad , \quad g(z) \neq 0 \ \forall z \in D_R(z_0).$$

$$d_n = \frac{(f/g)^{(n)}(z_0)}{n!}.$$

—

Example.

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots$$

For $|z| < \frac{\pi}{2}$, $\tan z = \frac{\sin z}{\cos z} = \dfrac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots}{1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots}$

$$\tan z = \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right)\left[1 + \left(\frac{z^2}{2!} - \frac{z^4}{4!} \cdots\right) + (\quad)^2 + \cdots\right]$$

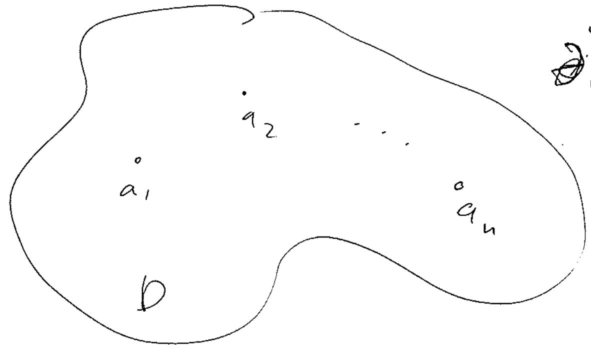
$$= z + \frac{z^3}{3} + \cdots$$

Chapter 6. Residues and Poles

in this chapter, we can compute

$$\int_{\partial D} f(z)\,dz = 2\pi i \sum_{j=1}^{n} \operatorname{Res}_{a_j}(f)$$

Residue of f at a_j



$f \in Hol(D \setminus \{a_1,..,a_n\})$.

68. Isolated singular points

Def. A singular point z_0 of f is isolated if $\exists \varepsilon > 0$ s.t. f is hol./analytic in $\{0 < |z - z_0| < \varepsilon\} = \dot{D}_\varepsilon(z_0)$.

Ex 1) $f = \dfrac{1}{(z-a)^n}$, $z = a$ is the isolated singular pt of f

$z = a$: (pole of f.)

2) $f = e^{\frac{1}{z}}$, $z = 0$ is the isolated singular pt of f.

3) $\operatorname{Log} z = \log|z| + i \operatorname{Arg} z$, $0 < \operatorname{Arg} z < 2\pi$

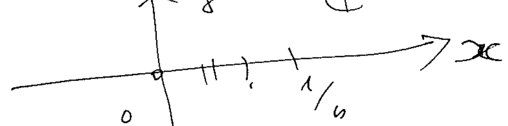
0 is a singular point of $\operatorname{Log} z$ but it is not isolated since $\operatorname{Log} z$ is not defined in $\dot{D}_\varepsilon(z_0)$ $(\forall\ \varepsilon > 0)$.

4) $f(z) = \begin{cases} \dfrac{\sin z}{z} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0. \end{cases}$, $f \in Hol(\Delta)$

but $g(z) = \dfrac{\sin z}{z}$ has an isolated singular pt $z = 0$.

5) $f(z) = \dfrac{1}{\sin \frac{\pi}{z}}$ has the singular pts $z = 0, \frac{1}{n}$, $n = 1, 2, \dots$

$z_n = \frac{1}{n}$ is isolated but $z = 0$ is not isolated

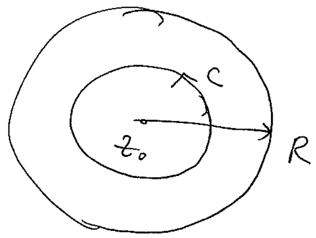


1

Def. f is said to have an isolated singular point at $z=\infty$ if $f \in Hol(\{R < |z| < \infty\})$ $(\exists R > 0)$.

69. **Residues** $\quad f \in Hol(\mathring{D}_R(z_0))$

- $$f(z) := \sum_{n=-\infty}^{+\infty} c_n (z-z_0)^n \quad (\text{Laurent series})$$
- C is a ~~single~~ simple closed contour
$C \subset D_R(z_0)$, $z=z_0 \in$ interior of C
C : is positively oriented



Then
$$\int_C f(z)\,dz = \sum_{n=-\infty}^{+\infty} c_n \int_C (z-z_0)^n\,dz = 2\pi i\, c_{-1}$$

$$c_{-1} = \frac{1}{2\pi i}\int_C f(z)\,dz \left(= \frac{1}{2\pi i} \oint_{|z-z_0|=\rho} f(z)\,dz \right), \quad 0<\rho<R$$

Def. $$\operatorname*{Res}_{z=z_0}(f) := \frac{1}{2\pi i}\oint_C f(z)\,dz$$

Examples

1) $f = e^{\frac{1}{z}}$, $\quad \operatorname*{Res}_{z=0} f = c_{-1} = 1$

2) $$f = \frac{1}{z^2+3z+2} = \frac{1}{(z+1)(z+2)}$$

$$\operatorname*{Res}_{z=-1} f = \frac{1}{1} = 1, \quad \operatorname*{Res}_{z=-2}(f) = -1.$$

70. Cauchy's residue theorem

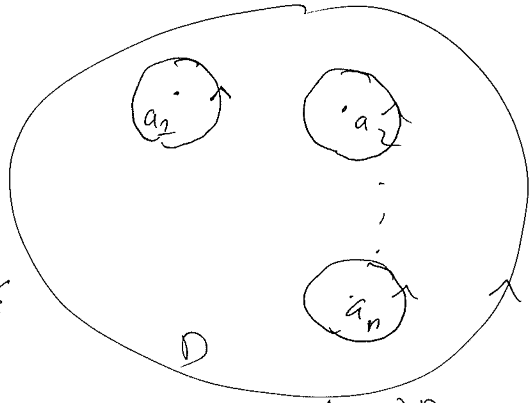
<u>Theorem</u>. Let C be a simple closed contour, positively oriented & Let D be a domain bounded by C. If $f \in Hol(\overline{D} \setminus \{a_1, .., a_n\})$, then

$$\int_C f(z)\,dz = 2\pi i \sum_{j=1}^{n} \operatorname{Res}_{z=a_j} f(z).$$

<u>Proof</u>. $\varepsilon > 0$

let $C_k := \{ |z-a_k| = \varepsilon \}$
$(k = 1, .., n)$, $C_k \subset D$

Then, by the Cauchy-Goursat's theorem we obtain that



$C = \partial D$

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \oint_{C_k} f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}_{z=a_k} f(z).$$

$\boxtimes$

<u>Examples</u>.

1) $$\oint_{|z|=5} \frac{dz}{z^2+3z+2} = 2\pi i \operatorname{Res}_{-1} f + 2\pi i \operatorname{Res}_{-2} f = 2\pi i \left(\frac{1}{+1} + \frac{1}{-1} \right) = 0$$

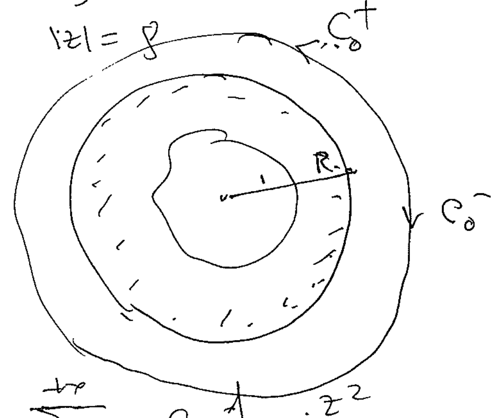
2) $$\int_{|z|=3} z^2 e^{1/z}\,dz = 2\pi i \operatorname{Res}_0 \left(z^2 e^{\frac{1}{z}}\right) = 2\pi i \cdot \frac{1}{3!} = \frac{\pi i}{3}.$$

§1 Residue at infinity

Def. Suppose that $f \in \mathrm{Hol}(\{R < |z| < +\infty\})$.

$$\operatorname*{Res}_{z=\infty} f(z) := \frac{1}{2\pi i}\int_{C_0^-} f(z)\,dz = -\frac{1}{2\pi i}\oint_{|z|=\rho} f(z)\,dz, \quad R < \rho.$$

: the residue of f at ∞.



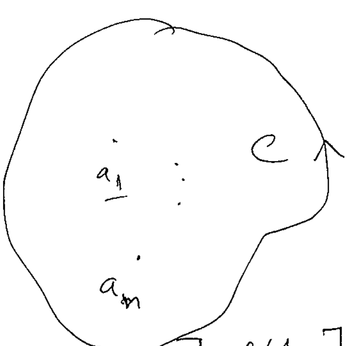
Remark

1) $f \in \mathrm{Hol}(\{R < |z| < +\infty\})$

$$f(z) = \sum_{n=-\infty}^{+\infty} c_n z^n, \quad \frac{1}{z^2} f\left(\frac{1}{z}\right) = \sum_{n=-\infty}^{+\infty} c_n \frac{1}{z^n} \cdot z^{2}$$

$$\operatorname*{Res}_{z=\infty} f(z) = \frac{1}{2\pi i}\int_{C_0^-} f(z)\,dz = -\frac{1}{2\pi i}\int_{C_0^+} f(z)\,dz = -c_{-1}.$$

$$= -\operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]$$



2) $f \in \mathrm{Hol}(\mathbb{C} \setminus \{a_1, .., a_n\})$

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=a_k} f = -2\pi i \operatorname*{Res}_{z=\infty} f = 2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]$$

$$\Rightarrow \quad \operatorname*{Res}_{\infty} f + \sum_{k=1}^{n} \operatorname*{Res}_{z=a_k} f = 0$$

Theorem 1. If $f \in \mathrm{Hol}(\mathbb{C} \setminus \{a_1, .., a_n\})$ and C is a positively oriented simple closed contour surrounding $a_1, .., a_n$, then

$$\int_C f(z)\,dz = 2\pi i \operatorname{Res}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right].$$

3) is

3) if f is holomorphic at ∞,

$$z\, f = z \sum_{n=-\infty}^{0} a_n z^n = z \sum_{n=0}^{\infty} a_n \cdot \frac{1}{z^n}$$

$$\operatorname{Res}_{\infty} f = -c_{-1} = -\lim_{z\to\infty} \left[z f(z) \right].$$

Example

1) $$f = \frac{1}{z^{2013}+1}$$

$$\int_{|z|=2} \frac{dz}{z^{2013}+1} = -2\pi i \operatorname{Res}_{\infty} f = -2\pi i \lim_{z\to\infty} \frac{z}{z^{2013}+1} = 0.$$

2) $$f = \frac{z+1}{z^2+3z+2}$$

$$\int_{|z|=3} \frac{(z+1)\,dz}{z^2+3z+2} = 2\pi i \operatorname{Res}_0 \left[\frac{1}{z^2} \left(\frac{\frac{1}{z}+1}{\frac{1}{z^2}+\frac{3}{z}+2} \right) \right]$$

$$= 2\pi i \operatorname{Res}_0 \left[\frac{1+z}{z(1+3z+2z^2)} \right] = 2\pi i \cdot \overset{1}{\cancel{\tfrac{2}{6}}} = \frac{2\pi i \cdot 1}{\cancel{3}}$$

$$= 2\pi i$$

3) $$\int_{|z|=R\gg 1} \frac{P_n(z)}{Q_m(z)}\, dz = 0 \quad \text{if} \quad m \geq n+2.$$

72. The 3 types of isolated singular points

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \underbrace{\frac{b_1}{z-z_0} + \frac{b_2}{\cdots} \cdots \frac{b_m}{(z-z_0)^m} + \cdots}_{\text{the principal part of } f \text{ at } z_0}$$

$$0 < |z-z_0| < R$$

1) $b_1 = \cdots = b_m = \cdots = 0$ (~~the~~ has no principal part)

z_0 is called a removable singular point of f

Ex $f(z) = \frac{\sin z}{z} = \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots}{z}$, $z \neq 0$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots$$

$z=0$ is a removable singular pt of $\frac{\sin z}{z}$.

Note $g(z) = \begin{cases} \frac{\sin z}{z} & z \neq 0 \\ 0 & z = 0 \end{cases}$ is holo. at $z=0$.

2) $\exists\, m \in \mathbb{N}^*$ s.t. $b_m \neq 0$, $b_{m+1} = \cdots = b_{m+k} = \cdots = 0$

- z_0 is called a pole of order m, m = order of f at z_0.
- $m = 1$, z_0 : simple pole

Ex $f(z) = \frac{\sin z}{z^6} = \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots}{z^6}$

$$= \frac{1}{z^5} - \frac{1}{3!\,z^3} + \frac{1}{5!\,z} + \cdots$$

- $z=0$: pole of order 5
- $\operatorname{Res}_0 f = \frac{1}{5!}$

3) If an infinite number of b_m are nonzero ($b_m \neq 0$ for an infinite number m)

then z_0 is called an essential singular point of f.

Ex

1) $f = e^{\frac{1}{z}} = 1 + \frac{1}{z} + \cdots + \frac{1}{n!\, z^n} + \cdots$

∘ $z = 0$ is an essential ———

∘ ~~$z = \infty$ is also~~ ———

2) $f = \sin \frac{1}{z} = \frac{1}{z} - \frac{1}{3!\, z^3} + \cdots$

$z = 0$ is the essential singular pt of f.

73. Residues at poles

Theorem. z_0 is a pole of order m $(m \geq 1)$ $\Longleftrightarrow$

$$f(z) = \frac{\phi(z)}{(z - z_0)^m}, \text{ where } \phi \text{ is hol at } z_0 \ \& \ \phi(z_0) \neq 0.$$

Moreover, $\operatorname*{Res}_{z = z_0} f(z) = \phi(z_0)$ if $m = 1$

$$\operatorname*{Res}_{z = z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!} \text{ if } m \geq 2.$$

Proof

"$\Rightarrow$"
$$f = \frac{\overset{\neq 0}{c_{-m}}}{(z - z_0)^m} + \frac{c_{-m+1}}{(z - z_0)^{m-1}} + \cdots + \frac{c_{-1}}{z - z_0} + c_0 + c_1 (z - z_0) + \cdots$$

$$= \frac{1}{(z - z_0)^m} \Big[\underbrace{c_{-m} + c_{-m+1}(z - z_0) + \cdots + c_0 (z - z_0)^m + \cdots}_{\phi(z)} \Big]$$

$$= \frac{\phi(z)}{(z - z_0)^m}$$

$\phi(z) = c_{-m} + c_{-m+1}(z - z_0) + \cdots$ is hol at $z = z_0$

& $\phi(z_0) = c_{-m} \neq 0$. $\quad \frac{\phi^{(m-1)}(z_0)}{(m-1)!} = c_{-1} = \operatorname*{Res}_{z_0} f$.

"$\Leftarrow$" Res $f = \dfrac{\phi(z)}{(z-z_0)^m}$, $\phi(z_0) \neq 0$

$$f = \frac{1}{(z-z_0)^m}\left[\phi(z_0) + \frac{\phi'(z_0)}{1!}(z-z_0) + \cdots + \frac{\phi^{(m)}(z_0)}{m!}(z-z_0)^m + \cdots\right]$$

$$= \frac{\phi(z_0)}{(z-z_0)^m} + \frac{\phi'(z_0)}{1!\,(z-z_0)^{m-1}} + \cdots$$

$\Rightarrow$ z_0 is a pole of order m of f & $c_{-1} = \dfrac{\phi^{(m-1)}(z_0)}{(m-1)!}$ $\boxtimes$

74 <u>Examples</u>

1) $$f(z) = \frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)}$$

$$= \frac{\frac{1}{z+i}}{z-i}$$

$\operatorname{Res}_{-i}(f) = \dfrac{1}{-i-i} = \dfrac{-1}{2i} = \dfrac{i}{2}$

$\operatorname{Res}_{+i}(f) = -\dfrac{i}{2}$

2) $f(z) = \dfrac{\log(1+z)}{z^2}$, $\operatorname{Res}_0 f = \left.\dfrac{1}{1+z}\right|_{z=0} = 1$.

3) $f = \dfrac{\cancel{\cos z}}{z^4}$, $\operatorname{Res}_0 f = \dfrac{(\cos z)^{(3)}(0)}{3!} = \dfrac{\sin(0)}{3!} = 0$

4) $$f = \frac{1}{e^z-1} - \frac{1}{z}$$

75. <u>Zeros of analytic functions</u>.

f is analytic at z_0

Def. z_0 is a zero of order $m \overset{def}{\Longleftrightarrow} f(z_0) = f'(z_0) = \dots = f^{(m-1)}(z_0) = 0$
$, f^{(m)}(z_0) \neq 0$.

<u>Theorem 1</u>. z_0 is a zero of order $m \Longleftrightarrow f(z) = (z-z_0)^m g(z)$
, where g is analytic at z_0 & $g(z_0) \neq 0$.

Proof.

<u>use</u>: $f(z) = f^{(m)}(z_0)\dfrac{(z-z_0)^m}{m!} + \dots$

$$= (z-z_0)^m \underbrace{\left[\frac{f^{(m)}(z_0)}{m!} + \frac{f^{(m+1)}(z_0)}{(m+1)!} + \dots \right]}_{g(z)}$$

$$= (z-z_0)^m g(z).$$

<u>Theorem 2</u>. Suppose that 1) f is analytic at z_0

2) $f(z_0) = 0$, $f \not\equiv 0$ in $D_\varepsilon(z_0)$ for any $\varepsilon > 0$.

Then $\exists \varepsilon > 0$ s.t. $f(z) \neq 0 \ \forall z \in \mathring{D}_\varepsilon(z_0)$ for a $\varepsilon > 0$..

Proof $f(z) = (z-z_0)^m g(z)$, $g(z_0) \neq 0$ (by Thm 1)

Since $g(z_0) \neq 0$, $\exists \varepsilon > 0$ s.t. $g(z) \neq 0 \ \forall z \in D_\varepsilon(z_0)$.

$\Rightarrow f(z) = \underbrace{(z-z_0)^m}_{\neq 0} g(z) \neq 0 \ \forall z \in \mathring{D}_\varepsilon(z_0)$ ☒

<u>Theorem 3</u>. Suppose that

a) $f \in Hol(N_0)$, N_0 is a nbhd of z_0

b) $f(z) \equiv 0$ on D or L, where $D \ni z_0$ is a domain

$D, L \subset N_0$ $z_0 \in L$ is a line segment.

Then $f(z) \equiv 0$ on/in N_0.

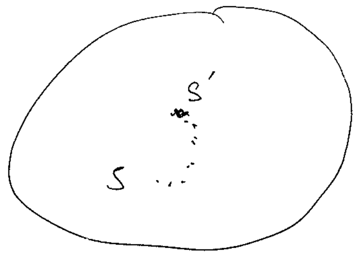
Proof (it follows from Thm 2).

Theorem 3'. $f \in Hol(D)$, $D \subset \mathbb{C}$ a domain.

Let $S \subset D$ satisfying $S' \cap D \neq \phi$

S': the set of accumulation pt of S.

~~If~~ If $f(z) \equiv 0$ in S, then $f \equiv 0$ on/in D.



Proof

$$A := \text{int}\{z \in D \mid f(z) = 0\}.$$

~~1)~~ A ~~is open since A ⊃ S~~

1) A is closed since if $z_n \to z_0$, $z_n \in S \Rightarrow f(z_n) = 0$

$\Rightarrow f(z_0) = 0$ ~~$\Rightarrow z_0 \in A$~~

3) ~~A is~~ $\xrightarrow{\text{by Thm 2}} f(z) \equiv 0$ in a nbd of z_0

$\Rightarrow z_0 \in A$.

2) A is open

3) $A \neq \phi$. Since $S' \cap D \neq \phi$, $\exists\, z_0 \in S' \cap D$

& $\{z_n\} \to z_0$ s.t. $f(z_n) = 0$ & $f(z_0) = 0$.

$\xrightarrow{\text{by Thm 2}} f(z) \equiv 0$ in a nbd of $z_0 \Rightarrow z_0 \in A$.

Thus A is nonempty, closed, and open. Therefore,

$A = D$ (since D is connected).

Zeros and Poles

Thm 1. Suppose that

a) p, q are analytic at z_0

b) $p(z_0) \neq 0$, q has a zero of order m at z_0

Then $\frac{p(z)}{q(z)}$ has a pole of order m at z_0

Proof. $q(z) = (z - z_0)^m \cdot g(z)$, $g(z_0) \neq 0$

$$\frac{p(z)}{q(z)} = \frac{p(z)/g(z)}{(z-z_0)^m} \quad ; \quad \frac{p}{g} \text{ is analytic at } z_0 \ \& \ \frac{p(z_0)}{g(z_0)} \neq 0$$

by Thm $\Rightarrow$ $\frac{p}{q}$ has a pole of order m at z_0

Thm 2. p, q are analytic at z_0. If

$p(z_0) \neq 0$, $q(z_0) = 0$, $q'(z_0) \neq 0$,

then z_0 is a simple pole of p/q and

$$\operatorname{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

Proof. $q(z) = (z - z_0) \cdot g(z)$, $g(z_0) \neq 0$

$$\frac{p(z)}{q(z)} = \frac{p(z)/g(z)}{z - z_0}, \quad \frac{p(z_0)}{g(z_0)} \neq 0$$

$$\operatorname{Res}_{z=z_0} \left(\frac{p(z)}{q(z)} \right) = \frac{p(z_0)}{g(z_0)} = \frac{p(z_0)}{q'(z_0)}.$$

Example

1) $f(z) = \tan z = \frac{\sin z}{\cos z}$, $z = \pi/2$

$\sin \frac{\pi}{2} = 1 \neq 0$, $\cos(\pi/2) = 0$, $\cos'(z) = -1 \neq 0$

$$\operatorname{Res}_{\pi/2} \tan z = \left. \frac{\sin z}{-\sin z} \right|_{z=z_0} = -1.$$

2) $f(z) = \dfrac{1}{z^4+1}$, $z_k = e^{i\frac{2k\pi}{4}} = e^{i\frac{k\pi}{2}}$, $k=0,1,2,3$.

$$\operatorname{Res}_{z_k} f(z) = \frac{1}{4z_k^3} = -\frac{z_k}{4}.$$

77. Behavior of functions near isolated singular points

<u>Theorem 1</u>. If z_0 is a pole of f, then $\lim_{z\to z_0} f(z) = \infty$

<u>Proof</u>

z_0 is a pole of $f \iff \exists m$ s.t $f(z) = \dfrac{\phi(z)}{(z-z_0)^m}$, $m \geq 1$.
$\phi(z_0) \neq 0$

$$\Rightarrow \lim_{z\to z_0} f(z) = \infty.$$

<u>Theorem 2</u>. If z_0 is a removable singular point of f, then f is analytic at z_0 and bounded in $\mathring{D}_{\varepsilon_0}(z_0)$.
(f can extend to be hol. at z_0)

<u>Proof</u> • $f(z) = a_0 + a_1(z-z_0) + \cdots$, $z \neq z_0$, $0<|z-z_0|<R_0$

Define $f(z_0) = a_0 \Rightarrow f(z) = a_0 + a_1(z-z_0) + \cdots \quad \forall z \in D_{R_0}(z_0)$

$\Rightarrow f \in Hol(D_{R_0}(z_0))$. For $0 < \varepsilon_0 < R$,

$f \in C^0(\overline{D_{\varepsilon_0}(z_0)}) \Rightarrow f$ is bounded in $\mathring{D}_{\varepsilon_0}(z_0)$. $\boxtimes$

<u>Lemma</u> (Riemann's theorem) $f \in Hol(\mathring{D}_\varepsilon(z_0))$ & ~~if~~ f is bounded on $\mathring{D}_\varepsilon(z_0)$. If f is <u>not</u> analytic at z_0, then it has a removable singularity there.

<u>Proof</u> • $f \in Hol(\mathring{D}_\varepsilon(z_0))$, $\exists M>0$ s.t. $|f(z)| \leq M$ ~~$\forall z \in$~~ $\mathring{D}_\varepsilon(z_0)$

• $f(z) = \displaystyle\sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{m=1}^{\infty} \frac{b_m}{(z-z_0)^m}$

$$b_m = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{f(z)}{(z-z_0)^{-m+1}}\,dz = \frac{1}{2\pi i}\int_{|z-z_0|=\rho} \frac{f(z)}{(z-z_0)^{-m+1}}\,dz$$

$0<\rho<\varepsilon$

$$0 \le |b_m| \le \frac{1}{2\pi}\cdot\frac{M\cdot 2\pi\rho}{\rho^{-m+1}} = \cancel{\tfrac{2\pi}{2\pi}}\, M\cdot\rho^m \xrightarrow{\rho\to 0^+} 0$$

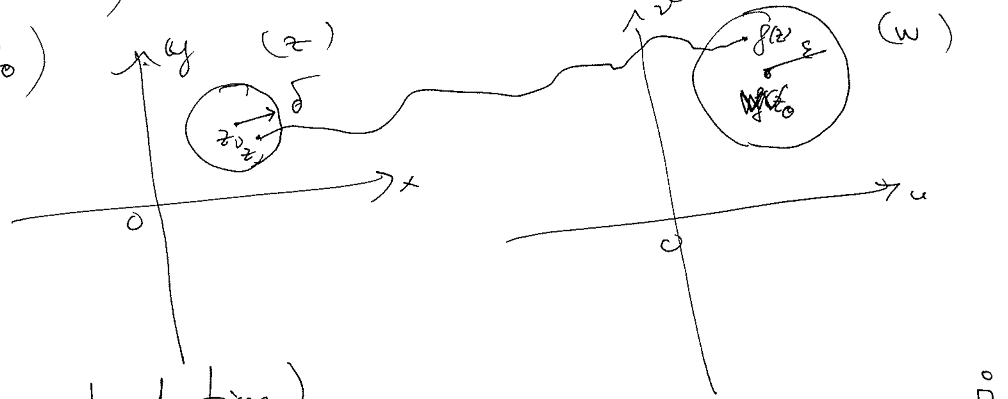
$\Rightarrow \quad b_m = 0 \quad \forall m = 1, 2, \dots$

Thus z_0 is a removable singular point of f.

Theorem 3: z_0 is an essential singular point of f.
$w_0 \in \mathbb{C}$, $\varepsilon > 0$.

Then $\forall \delta>0$, $\exists z \in \dot{D}_\delta(z_0)$ s.t. $|f(z) - w_0| < \varepsilon$.
$(0<\delta<\varepsilon_0)$

i.e. $\overline{f(\dot{D}_\delta(z_0))} = \overline{\mathbb{C}}$



Proof (by contradiction)

Suppose that $\exists\ \delta>0$ s.t. $|f(z) - w_0| \ge \varepsilon \quad \forall z \in \dot{D}_\delta(z_0)$

$$g(z) := \frac{1}{f(z)-w_0}, \qquad 0<|z-z_0|<\delta$$

$g(z) \in Hol(\dot{D}_\delta(z_0))$ and $|g(z)| = \frac{1}{|f(z)-w_0|} \le \frac{1}{\varepsilon} < +\infty$

by lemma $\Rightarrow$ z_0 is a removable singular point of g & g is analytic at z_0 $(g(z_0) := a_0)$.

+ $g(z_0) \neq 0 \Rightarrow \quad f(z) = \dfrac{1}{g(z)} + w_0$ is analytic at z_0

$$f(z_0) := \frac{1}{g(z_0)} + w_0$$

$\Rightarrow$ z_0 is a removable singular point of f. ><

+ $g(z_0) = 0 \Rightarrow$ z_0 is zero (of g) of order $m \geq 1$

$$g(z) = (z-z_0)^m h(z) \quad , \quad h(z_0) \neq 0$$

$$\Rightarrow \quad f(z) = \frac{1/h(z_0)}{(z-z_0)^m} + w_0$$

$\Rightarrow$ z_0 is a pole of order m ><.

Thus the proof is complete.

Chapter 7. Application of Residues

- $\displaystyle\int_{-\infty}^{+\infty} \frac{P_n(x)}{Q_m(x)}\,dx \quad (m \geq n+2)$
- $\displaystyle\int_{-\infty}^{+\infty} \frac{P_n(x)}{Q_m(x)}\cos(\alpha x)\,dx \;,\; \int_{-\infty}^{+\infty} \frac{P_n(x)}{Q_m(x)}\sin(\alpha x)\,dx \quad (\alpha > 0)$
- $\displaystyle\int_{0}^{2\pi} F(\sin\theta, \cos\theta)\,d\theta$

78. Evaluation of improper integrals

Recall $\quad f : [a, +\infty) \longrightarrow \mathbb{R} \;,\; g : (-\infty, +\infty) \longrightarrow \mathbb{R}$

- $\displaystyle\int_a^{+\infty} f(x)\,dx := \lim_{R\to+\infty} \int_a^R f(x)\,dx$ (converges)
- $\displaystyle\int_{-\infty}^{+\infty} g(x)\,dx = \lim_{R_1\to+\infty} \int_{-R}^{0} f(x)\,dx + \lim_{R_2\to+\infty} \int_0^{R_2} f(x)\,dx .$

Ex. 1) $\displaystyle\int_0^{+\infty} \frac{dx}{x^2+1} = \arctan x \Big|_0^{+\infty} = \frac{\pi}{2}$

2) $\displaystyle\int_{-\infty}^{+\infty} \frac{dx}{x}$ is not convergent

$$\text{P.V}\int_{-\infty}^{+\infty} f(x)dx := \lim_{R\to+\infty} \int_{-R}^{R} \frac{dx}{x} = 0$$

Def. $\displaystyle \text{P.V.} \int_{-\infty}^{+\infty} f(x)\,dx = \overset{\exists}{\lim_{R\to+\infty}} \int_{-R}^{R} f(x)\,dx$: Cauchy principal value

Ex. $\displaystyle \text{P.V.} \int_{-\infty}^{+\infty} x^3\,dx = 0$; if $\displaystyle\int_{-\infty}^{+\infty} f(x)\,dx$ converges, then $\displaystyle \text{P.V.}\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{+\infty} f(x)\,dx .$

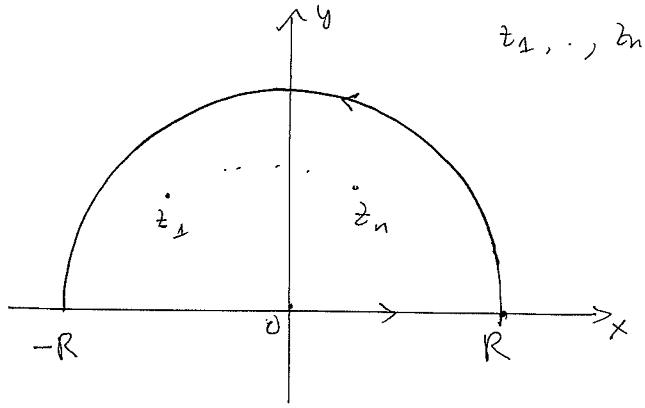
- $f(z) = \dfrac{p(z)}{q(z)}$: rational ~~number~~ function, $q(x)$ has no real zeros

$\{q(z)=0\} \Rightarrow \{z_1, \dots, z_n\}$, $\operatorname{Im} z_j > 0$.

- $\exists$ P.V. $\displaystyle\int_{-\infty}^{+\infty} f(x)\,dx$

$R >> 1$ s.t.

$z_1, \dots, z_n \in D_R$



$\Gamma(R) := [-R, R] \cup C_R$, $C_R = \{|z| = R, \operatorname{Im} z \geq 0\}$

$= \partial D_R$.

By Cauchy's residue theorem, $\displaystyle\int_{\Gamma(R)} f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}_{z=z_k} f(z)$

$$\Leftrightarrow \int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}_{z=z_k} f(z)$$

$\downarrow$ $\qquad \downarrow$ if $\deg q \geq \deg p + 2$

$\qquad\quad 0$

$$\text{P.V.} \int_{-\infty}^{+\infty} f(x)\,dx = 2\pi i \sum_{k=1}^{n} \operatorname{Res}_{z_k} f(z)$$

Theorem. If $\deg q \geq \deg p + 2$, then

$$\text{P.V.} \int_{-\infty}^{+\infty} f(x)\,dx = 2\pi i \sum_{k=1}^{n} \operatorname{Res}_{z_k} f(z).$$

Proof. $\displaystyle\frac{p(z)}{q(z)} = \frac{a_m z^m + \cdots + a_1 z_1 + a_0}{b_k z^k + \cdots + b_0} \sim \frac{\text{const}}{z^{k-m}}$.

as $z \to \infty$.

$$\Rightarrow \left|\frac{p(z)}{q(z)}\right| \leq \frac{|\text{const}|}{R^{k-m}} \leq \frac{|\text{const}|}{R^2} = \frac{M}{R^2}.$$

$$\left|\int_{C_R} f(z)\,dz\right| \leq \frac{M}{R^2}\cdot \pi R = \frac{\pi M}{R} \longrightarrow 0 \quad \text{as } R\to+\infty.$$

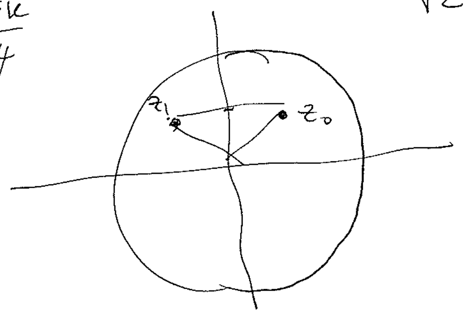
Thus $\int_{C_R} f(z)\,dz \to 0$ as $R \to +\infty$ $\boxtimes$

79. <u>Examples</u>

1) $$I = \int_{-\infty}^{+\infty} \frac{dx}{x^4+1} = 2\pi i\left[\operatorname{Res}_{z_0} f + \operatorname{Res}_{z_1} f(z)\right] = -2\pi i\,\frac{(z_0+z_1)}{4} = -2\pi i\,\frac{2\, i\frac{\sqrt{2}}{2}}{4} = \frac{\pi}{\sqrt{2}}.$$

$$z^4+1=0 \Rightarrow z_k = e^{i\frac{\pi+2k\pi}{4}},\ k=0,1,2,3,$$

$$\operatorname{Res}_{z_k} f(z) = \frac{1}{4z_k^3} = -\frac{z_k}{4}$$



$$\int_0^{+\infty} \frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}.$$

2) $$I = \int_0^{+\infty} \frac{dx}{x^6+1} = \frac{1}{2}\left(2\pi i\left(-\frac{z_0}{6}-\frac{z_1}{6}-\frac{z_2}{6}\right)\right) = \frac{2\pi i(-2i)}{2\cdot 6} = \frac{\pi}{3}.$$

$$z^6+1=0 \Rightarrow z_k = e^{i\frac{\pi+2k\pi}{6}},\ k=0,1,2,3,4,5$$

$$z_0+z_1+z_2 = 2i\sin\frac{\pi}{6} + i = 2i\frac{1}{2}+i = 2i$$

$$\operatorname{Res}_{z=z_k}\left(\frac{1}{z^6+1}\right) = -\frac{z_k}{6}.$$

$$\left|\int_{C_R}\frac{dz}{z^6+1}\right| \leq \frac{1}{R^6-1}\cdot \pi R \longrightarrow 0 \quad \text{as } R\to+\infty.$$

80. Improper integrals from Fourier Analysis.

$$\int_{-\infty}^{+\infty} f(x)\sin(ax)\,dx\,, \quad \int_{-\infty}^{+\infty} f(x)\cos(ax)\,dx\,, \quad a>0.$$

$$\int_{-\infty}^{+\infty} f(x)\sin(ax)\,dx = \operatorname{Im}\int_{-\infty}^{+\infty} f(x)\,e^{iax}\,dx$$

$$\int_{-\infty}^{+\infty} f(x)\cos(ax)\,dx = \operatorname{Re}\int_{-\infty}^{+\infty} f(x)\,e^{iax}\,dx$$

$$\left|e^{iax}\right| = \sqrt{\cos^2(ax)+\sin^2(ax)} = 1$$

$$\left|e^{iaz}\right| = \left|e^{ia(x+iy)}\right| = e^{-ay} \leq 1 \text{ if } y \geq 0.$$

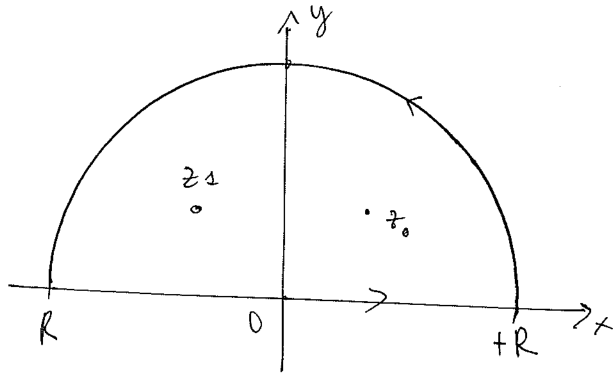
(circumference $:= 2\pi R$)

Example

$$I = \int_{-\infty}^{+\infty} \frac{\cos(2x)}{x^4+1}\,dx = \operatorname{Re}\int_{-\infty}^{+\infty} \frac{e^{2ix}}{x^4+1}\,dx = \operatorname{Re} J$$

$$f(z) := \frac{e^{2iz}}{z^4+1}\,, \quad \operatorname{Res}_{z_k} f(z) = \frac{e^{2iz_k}}{4z_k^3} = -\frac{z_k e^{2iz_k}}{4}$$

$$\Gamma(R) = [-R,R] \cup C_R$$



$$\underbrace{\int_{-R}^{R} f(x)\,dx}_{\downarrow \atop J} + \underbrace{\int_{C_R} f(z)\,dz}_{\downarrow \atop 0} = 2\pi i\left[\operatorname{Res}_{z_0} f(z) + \operatorname{Res}_{z_1} f(z)\right]$$

$$0 \le \left| \int_{C_R} \frac{e^{2iz}}{z^4+1}\, dz \right| \le \frac{1}{R^4-1} \cdot \pi R \longrightarrow 0$$

$$\to 0$$

$$J = \frac{2\pi i}{4}\left(- e^{i\frac{\pi}{4}} \cdot e^{2i\cdot e^{i\frac{\pi}{4}}} - e^{i\frac{3\pi}{4}}\, e^{2i e^{i\frac{3\pi}{4}}} \right)$$

$$= -\frac{\pi i}{2}\left(e^{i\left(\frac{\pi}{4} + 2e^{i\frac{\pi}{4}}\right)} - e^{i\left(\frac{3\pi}{4} + 2e^{i\frac{3\pi}{4}}\right)} \right)$$

$$I = \operatorname{Re} J = \frac{\pi}{2}\left(\operatorname{Im} e^{i\left(\frac{\pi}{4} + 2\cos\frac{\pi}{4} + 2i\sin\frac{\pi}{4}\right)} - \operatorname{Im} e^{i\left(\frac{3\pi}{4} + 2\cos\frac{3\pi}{4} + 2i\sin\frac{3\pi}{4}\right)} \right)$$

$$= \frac{\pi}{2} \operatorname{Im}\left(e^{i\left(\frac{\pi}{4} + \sqrt{2} + i\sqrt{2}\right)} - e^{i\left(\frac{3\pi}{4} + \sqrt{2} - i\sqrt{2}\right)} \right)$$

$$\int_{-\infty}^{+\infty} \frac{\cos 2x}{x^4+1}\, dx = \frac{\pi}{2}\left[e^{-\sqrt{2}} \sin\left(\frac{\pi}{4} + \sqrt{2}\right) - e^{\sqrt{2}} \sin\left(\frac{3\pi}{4} + \sqrt{2}\right) \right]$$

81 **Jordan's lemma**

Theorem (Jordan's lemma). Suppose that

a) f is holo/analytic in $\{ z \in \mathbb{C} \mid |z| > R_0,\ \operatorname{Im} z \ge 0 \}$

b) $C_R = \{ |z| = R,\ \operatorname{Im} z \ge 0 \}$, $R > R_0$

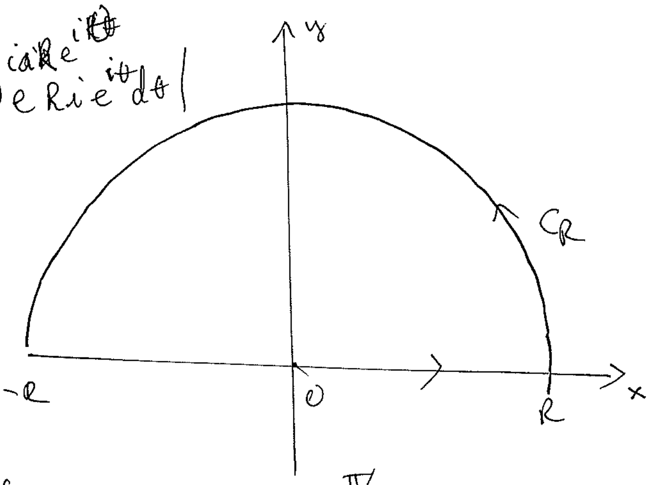
c) $\sup_{z \in C_R} |f(z)| \le M_R \longrightarrow 0$ as $R \to +\infty$

Then for $\forall a > 0$, we have

$$\lim_{R \to +\infty} \int_{C_R} f(z)\, e^{iaz}\, dz = 0$$

Proof

$$0 \le \left| \int_{C_R} f(z)\, e^{iaz}\, dz \right| = \left| \int_0^{\pi} f(Re^{i\theta})\, e^{iaRe^{i\theta}}\, Ri e^{i\theta}\, d\theta \right|$$



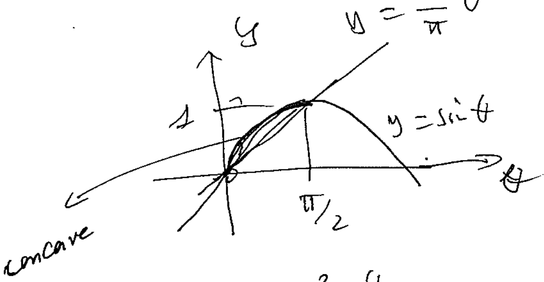
$e^{iRae^{i\theta}}$

$$\le M_R \int_0^{\pi} \left| e^{iaRe^{i\theta}} \right| R\, d\theta$$

$$\le M_R \cdot R \cdot \int_0^{\pi} e^{-R\sin\theta}\, d\theta = 2M_R \cdot R \int_0^{\pi/2} e^{-R\sin\theta}\, d\theta$$

$$\le 2M_R \cdot R \cdot \int_0^{\pi/2} e^{-\frac{2R}{\pi}\theta}\, d\theta$$

$$= 2M_R \cdot R \cdot \left. \frac{e^{-\frac{2R}{\pi}\theta}}{-\frac{2R}{\pi}} \right|_0^{\pi/2}$$



$\sin\theta \ge \frac{2}{\pi}\theta$

$\forall\ 0 \le \theta \le \pi/2$

$$0 \le \left| \int_{C_R} f(z)\, e^{iaz}\, dz \right| = 2M_R \cdot \frac{\pi}{2}\left(1 - e^{-R}\right) < \pi M_R \xrightarrow{R \to +\infty} 0$$

Thus $\int_{C_R} f(z)\, e^{iaz}\, dz \longrightarrow 0$ as $R \to +\infty$.

Example

1) $\int_{C_R} \frac{e^{iz}}{z}\, dz \longrightarrow 0$ as $R \to +\infty$

2) $\int_{C_R} \frac{e^{iz}}{z+1}\, dz \longrightarrow 0$ as $R \to +\infty$

Theorem. if $\deg q \geq \deg p + 1$, then

$$\text{P.V.} \int_{-\infty}^{+\infty} f(x)\, e^{iax}\, dx = 2\pi i \sum_{k=1}^{n} \operatorname{Res}_{z_k} \left[f(z)\, e^{iaz} \right]$$

— $q(z_k) = 0$, $\operatorname{Im} z_k > 0$.

Example

1) $$I = \int_0^{+\infty} \frac{x \sin 2x}{x^2+3}\, dx = \frac{1}{2} \operatorname{Im} \int_{-\infty}^{+\infty} \frac{x\, e^{2ix}}{x^{2}+3}\, dx = \frac{1}{2} \operatorname{Im} J.$$

$$f(z) = \frac{z\, e^{2iz}}{z^2+3}, \quad \operatorname{Res}_{z=i\sqrt{3}} f(z) = \left. \frac{z\, e^{2iz}}{2z} \right|_{z=i\sqrt{3}} = \frac{e^{-2\sqrt{3}}}{2}$$

$$\operatorname{Res}_{z=-i\sqrt{3}} f(z) = \quad = e^{2\sqrt{3}}.$$

$$\int_{C_R} \frac{z e^{2iz}}{z^2+3}\, dz \longrightarrow 0 \text{ as } R \to +\infty \text{ (by Jordan's lemma)}$$

$$\Rightarrow J = 2\pi i \operatorname{Res}_{z=i\sqrt{3}} f(z) = 2\pi i \frac{e^{-2\sqrt{3}}}{2} = \pi i\, e^{-2\sqrt{3}}$$

$$I = \frac{1}{2} \cdot \operatorname{Im} \left(\pi i\, e^{-2\sqrt{3}} \right) = \frac{\pi}{2} e^{-2\sqrt{3}} \quad \boxtimes.$$

§ 2) $$I = \int_{-\infty}^{+\infty} \frac{x \sin x}{x^2+1}\, dx$$

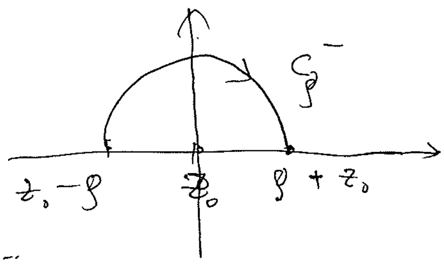
82. ## Indented Paths (Indented paths)

Theorem. Suppose that

a) f has a simple pole at $z = z_0$, $\operatorname{Res}_{z_0} f(z) = B_0$

b) $C_\rho^- = \{ |z - z_0| = \rho, \ \operatorname{Im}(z - z_0) \geq 0 \}$, $0 < \rho << 1$.

Then $\displaystyle\lim_{\rho \to 0^+} \int_{C_\rho^-} f(z)\, dz = -B_0 \pi i$



Proof

- $f(z) = \dfrac{B_0}{z - z_0} + a_0 + a_1(z - z_0) + \cdots$ $\quad 0 < |z - z_0| < R$

$= \dfrac{B_0}{z - z_0} + g(z)$, $\exists\, 0 < \rho_0 < R$ s.t

$g(z) \in Hol(\overline{D_{\rho_0}(z_0)})$, $M := \max_{|z - z_0| \leq \rho_0} |g(z)|$

- $\displaystyle\int_{C_\rho^-} f(z)\, dz = B_0 \int_{C_\rho^-} \frac{dz}{z - z_0} + \int_{C_\rho^-} g(z)\, dz$

$\displaystyle\int_{C_\rho^-} \frac{dz}{z - z_0} = -\int_0^\pi \frac{\rho i e^{i\theta}}{\rho e^{i\theta}}\, d\theta = -\pi i$ $\quad 0 < \rho < \rho_0 < R$

$\displaystyle\left|\int_{C_\rho^-} g(z)\, dz\right| \leq M \cdot \pi\rho \longrightarrow 0$ as $\rho \to 0^+$

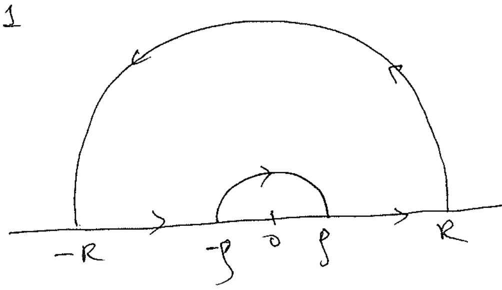
- $\Rightarrow \displaystyle\lim_{\rho \to 0^+} \int_{C_\rho^-} f(z)\, dz = -B_0 \pi i$ $\boxed{\times}$

Example

1) $I = \int_0^{+\infty} \frac{\sin x}{x}\,dx = \frac{1}{2}\operatorname{Im}\int_{-\infty}^{+\infty} \frac{e^{ix}}{x}\,dx = \frac{1}{2}\operatorname{Im} J$

$\Gamma(R,\rho) = [-R,-\rho] \cup [\rho, R] \cup C_R^+ \cup C_\rho^-, \quad 0<\rho<R<+\infty.$

$f(z) = \frac{e^{iz}}{z}, \quad \operatorname{Res}_0 f = e^{i\cdot 0} = 1$



$\int_{C_\rho^-} \frac{e^{iz}}{z}\,dz \xrightarrow{\rho\to 0^+} -\pi i$

Check $\quad \frac{e^{iz}}{z} = \frac{1 + iz + \frac{(iz)^2}{2!} + \cdots}{z}$

$= \frac{1}{z} + g(z)$

$\int_{C_\rho^-} \frac{e^{iz}}{z}\,dz = \int_{C_\rho^-} \frac{dz}{z} + \int_{C_\rho^-} g(z)\,dz \to -\pi i$

- $\int_{C_\rho} \frac{dz}{z} = -\pi i$
- $\left| \int_{C_\rho^-} g(z)\,dz \right| \leq \max_{z\in C_\rho^-} |g(z)| \cdot \pi\rho \to 0$ as $\rho \to 0^+$

Thus $\int_{C_\rho^-} \frac{e^{iz}}{z}\,dz \Rightarrow -\pi i$ as $\rho \to 0^+$

$J = \pi i$

$I = \frac{1}{2}\operatorname{Im}(\pi i)$

$= \frac{\pi}{2}$

By the Cauchy - Goursat theorem, we have $\to J$

$0 = \int_{\Gamma(R,\rho)} \frac{e^{iz}}{z}\,dz = \left[\int_{-R}^{-\rho} \frac{e^{iz}}{z}\,dz + \int_\rho^R \frac{e^{iz}}{z}\,dx\right] + \int_{C_R^+} f(z)\,dz + \int_{C_\rho^-} f(z)\,dx$

$z = -x \qquad dz = -dx$

$0 = \int_{\rho}^{R} \left[-\frac{e^{-ix}}{x} + \frac{e^{ix}}{x}\right] dx + \int_{C_R^+} f\,dz + \int_{C_\rho^-} f\,dz$

$\int_R^\rho \frac{e^{-ix}}{-x}(-dx) = \int_\rho^R -\frac{e^{-ix}}{x}\,dx$

$$0 = 2i \underbrace{\int_\rho^R \frac{\sin x}{x}\,dx}_{\downarrow\; 2i\,I} + \underbrace{\int_{C_R} f(z)\,dz}_{\downarrow\; 0} + \underbrace{\int_{C_\rho^-} f(z)\,dz}_{\downarrow\; -\pi i} \quad \Rightarrow 2I - \pi = 0$$

$$\Rightarrow I = \lim_{\substack{R\to+\infty \\ \rho\to 0^+}} \int_\rho^R \frac{\sin x}{x}\,dx = \frac{\pi}{2} \ \boxtimes$$

2) $$I = \int_0^{+\infty} \frac{1-\cos(x)}{x^2}\,dx = \frac{1}{2}\,\mathrm{Re}\int_{-\infty}^{+\infty} \frac{1-e^{ix}}{x^2}\,dx$$

$$f(z) = \frac{1-e^{iz}}{z^2} = \frac{\cancel{1} - (\cancel{1} + (iz) + \frac{(iz)^2}{2!} + \dots)}{z^2} = -\frac{i}{z} + g(z)$$

- $$\int_{C_\rho^-} f(z)\,dz \longrightarrow -(-i)\cdot\pi i = -\pi \text{ as } \rho \to 0^+$$

- $$\int_{C_R} f(z)\,dz = \underbrace{\int_{C_R} \frac{dz}{z^2}}_{\downarrow\; 0} - \underbrace{\int_{C_R} \frac{e^{iz}}{z^2}}_{\downarrow\; 0 \text{ by Jordan's lemma}} \longrightarrow 0 \text{ as } R\to+\infty$$

$$\left|\int_{C_R} \frac{dz}{z^2}\right| \le \frac{1}{R^2}\cdot\pi R \longrightarrow 0 \text{ as } R\to+\infty.$$

By the Cauchy-Goursat's theorem, we have

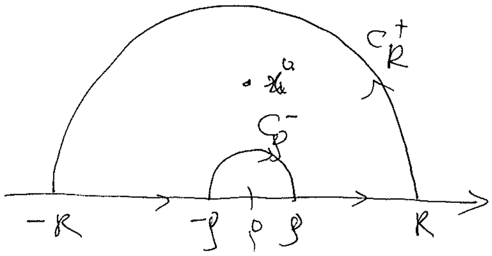
$$0 = \int_{\Gamma(R,\rho)} f(z)\,dz = \int_{C_R} f(z)\,dz + \int_{C_\rho^-} f(z)\,dz + \int_\rho^R f(x)\,dx + \int_{-R}^{-\rho} f(z)\,dz \qquad \begin{array}{l} z = -x \\ dz = -dx \end{array}$$

$$= \underbrace{\int_{C_R} f\,dz}_{\downarrow\; 0} + \underbrace{\int_{C_\rho} f\,dz}_{\downarrow\; -\pi} + \underbrace{\left[\int_\rho^R f(x)\,dx + \int_{-R}^{-\rho} \frac{1-e^{+ix}}{x^2}\,dx\right]}_{\downarrow\; \int_{-\infty}^{+\infty} \frac{1-e^{ix}}{x^2}\,dx} \Rightarrow \int_{-\infty}^{+\infty} \frac{1-e^{ix}}{x^2}\,dx = \pi$$

$$\Rightarrow I = \frac{\pi}{2} \ \boxtimes.$$

83. An ~~indentation~~ around a branch point (indentation)

Example 1.



$$\int_0^{+\infty} \frac{\ln x}{(x^2+1)^2} = -\frac{\pi}{4}.$$

$$f(z) = \frac{\operatorname{Log} z}{(z^2+1)^2}, \quad \operatorname{Log} z = \log|z| + i \operatorname{Arg} z$$
$$-\pi/2 < \operatorname{Arg} z < \frac{3\pi}{2}$$

$$\Gamma(R,\rho) = [-R,-\rho] \cup [\rho, R] \cup C_\rho^- \cup C_R^+$$

$$f(z) = \frac{\log z/(z+i)^2}{(z-i)^2}, \quad \operatorname{Res}_i f(z) = \left(\frac{\log z}{(z+i)^2}\right)'\Bigg|_{z=i} =$$

$$\operatorname{Res}_i f(z) = \frac{1}{z(z+i)^2} - \frac{2\log z}{(z+i)^3}\Bigg|_{z=i} = -\frac{1}{i4} + 2\cdot\frac{i\frac{\pi}{2}}{8i} = \frac{\pi}{8} + \frac{i}{4}$$

$$\left|\int_{C_\rho^-} \frac{\ln z}{(z^2+1)^2}\,dz\right| \le \frac{\sqrt{\log^2\rho + \pi^2}}{(1-\rho^2)^2}\cdot \pi\rho \to 0 \text{ as } \rho \to 0$$

$$\left|\int_{C_R} \frac{\ln z}{(z^2+1)^2}\,dz\right| \le \frac{\sqrt{\log^2 R + \pi^2}}{(R^2-1)^2}\cdot \pi R \to 0 \text{ as } R \to +\infty.$$

By the Cauchy's residue theorem, we get

$$2\pi i \operatorname{Res}_i f(z) = \int_{\Gamma(R,\rho)} f(z)\,dz = \int_{-R}^{-\rho} \frac{\ln z}{(z^2+1)^2}\,dz + \int_\rho^R f(x)\,dx + \underbrace{\int_{C_R^+} f(z)\,dz}_{\to 0} + \underbrace{\int_{C_\rho^-} f(z)\,dz}_{\to 0}$$

$$z = -x$$

$$\Leftrightarrow 2\pi i\left(\frac{\pi}{8} + \frac{i}{4}\right) = -\int_R^\rho \frac{\log x + \pi i}{(x^2+1)^2}\,dx + \int_\rho^R f(x)\,dx + \underline{\qquad\qquad}$$

$$\operatorname{Re}\left(\qquad\right) = \operatorname{Re}\left(\underline{\qquad\qquad\qquad}\right)$$

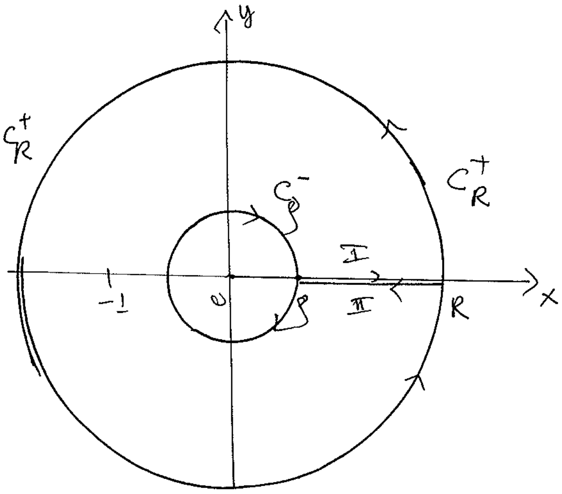
$$\downarrow$$

$$-\frac{\pi}{2} = 2\int_0^{+\infty} \frac{\ln x}{(x^2+1)^2}\,dx \Rightarrow \int_0^{+\infty} \frac{\ln x}{(x^2+1)^2} = -\frac{\pi}{4}$$

84. Integration along a branch cut

Ex $$I = \int_0^{+\infty} \frac{x^{-a}}{x+1}\,dx \qquad (0<a<1)$$

$0 < \rho < 1 < R < +\infty$



$\Gamma(R,\rho) = \underset{\text{I}}{[\rho,R]} \cup \underset{\text{II}}{[R,\rho]} \cup C_\rho^- \cup C_R^+$

$$f(z) = \frac{z^{-a}}{z+1}$$

$$z^{-a} = e^{-a\,\mathrm{Log}\,z}$$

$$= e^{-a(\ln|z| + i\,\mathrm{Arg}\,z)}, \quad 0 \le \mathrm{Arg}\,z \le 2\pi$$

$$\mathrm{Res}_{-1} f(z) = z^{-a}\Big|_{z=-1} = e^{-a(0+i\pi)} = e^{-a\pi i}$$

$$\int_{[R,\rho]} f(z)\,dz = \int_{(\text{II})} f(z)\,dz = \int_R^\rho \frac{e^{-a(\ln x + 2\pi i)}}{x+1}\,dx$$

$$= -\int_\rho^R \frac{x^{-a}\cdot e^{-2\pi a i}}{x+1}\,dx$$

By Cauchy's residue theorem,

$$2\pi i\,\mathrm{Res}_{-1} f(z) = \int_{\Gamma(R,\rho)} f(z)\,dz = \underbrace{\int_{C_\rho^-} f(z)\,dz}_{\to 0} + \underbrace{\int_{C_R^+} f(z)\,dz}_{\to 0} + \left(1 - e^{-2\pi a i}\right)\int_0^{+\infty} \frac{x^{-a}}{x+1}\,dx$$

$$\Rightarrow \quad I = 2\pi i \cdot \frac{e^{-ia\pi}}{1 - e^{-i2a\pi}} = \frac{2\pi i}{2i} \cdot \frac{1}{\frac{e^{i\pi a} - e^{-i\pi a}}{2i}}$$

$$= \frac{\pi}{\sin \pi a} \quad \boxed{\times}$$

Note: $\left|\int_{C_R} f(z)\,dz\right| \le \frac{R^{-a}}{R-1}\pi R \to 0$; $\left|\int_{C_\rho^-} f(z)\,dz\right| \le \frac{\rho^{-a}}{1-\rho}\cdot\pi\rho \to 0$

as $R \to +\infty$ & $\rho \to 0^+$

85. Definite integrals involving sines and cosines

$$I = \int_0^{2\pi} F(\sin\theta, \cos\theta)\, d\theta$$

$$z = e^{i\theta} \qquad 0 \leq \theta \leq 2\pi$$

$$dz = i e^{i\theta} d\theta \implies d\theta = \frac{dz}{iz}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z - z^{-1}}{2i} = \frac{z^2+1}{2z}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - z^{-1}}{2i} = \frac{z^2-1}{2iz}$$

$$I = \int_{|z|=1} F\left(\frac{z^2-1}{2iz}, \frac{z^2+1}{2z}\right) \cdot \frac{dz}{iz}.$$

Ex.
$$I = \int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta} \qquad (-1 < a < 1)$$

$$= \int_{|z|=1} \frac{\frac{dz}{iz}}{1 + a\frac{z^2+1}{2z}} = \frac{2}{i}\int_{|z|=1} \frac{dz}{az^2 + 2z + a}$$

$$az^2 + 2z + a = 0, \quad \Delta' = 1 - a^2, \quad \sqrt{\Delta} = \sqrt{1-a^2}$$

$$z_1 = \frac{-1 + \sqrt{1-a^2}}{a}, \quad z_2 = -1 - \sqrt{1-a^2} < -1$$

$$\operatorname{Res}_{z_1} \frac{1}{az^2 + 2z + a} = \frac{1}{2az + 2} = \frac{1}{2(-1 + \sqrt{1-a^2} + 1)} = \frac{1}{2\sqrt{1-a^2}}$$

$$I = \frac{2}{i} \cdot 2\pi i \operatorname{Res}_{z_1} \frac{1}{az^2 + 2z + a} = 4\pi \cdot \frac{1}{2\sqrt{1-a^2}} = \frac{2\pi}{\sqrt{1-a^2}} \quad \boxtimes.$$

$$w_0 \xrightarrow{w}$$

$$\Gamma \ni w: \; w_0 \longrightarrow w_0$$

$\arg w: \; \phi_0 \longrightarrow \phi_1$ is a value of $\arg w_0$

$$\Rightarrow \frac{1}{2\pi} \Delta_C \arg f(z) = \phi_1 - \phi_0 \in 2\pi \mathbb{Z}$$

When w returns to w_0, $\arg w$ is a value of $\arg w_0$, denote by ϕ_1

$$\frac{1}{2\pi i} \Delta_C \arg f(z) \overset{def}{:=} \tilde{\phi}_1 - \tilde{\phi}_0 \in \mathbb{Z}$$

86. ## Argument principle

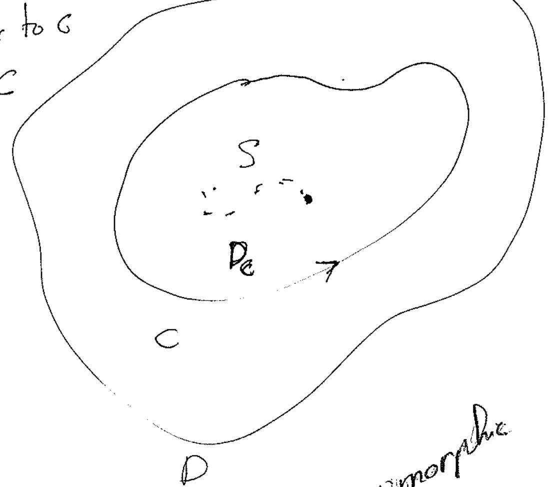
Def. $f: \mathbb{C} \supset \underset{\text{a domain}}{D} \to \mathbb{C}$ is meromorphic (in D) if it is holo. in D except for poles

Let C be a simple closed contour, positively oriented, $C \subset D$ s.t. $f(z) \neq 0 \ \forall z \in C$

Remark

1) # of zeros of f is finite inside C (interior to C)

Indeed,
$$S = \{ z \in D_C \mid f(z) = 0 \}$$
if $\# S = +\infty$, then $S' \cap D_C \neq \phi$
by Identity Theorem $\Rightarrow$ $f \equiv 0$ $>\!<$

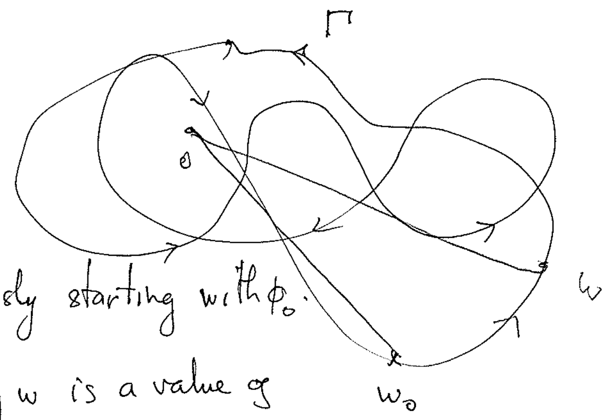


2) # of poles of f in D_C is finite

Indeed,
$$\# \text{ of poles of } f = \# \text{ of zeros of } \left(\frac{1}{f}\right) < +\infty \ \boxtimes$$
meromorphic

$\Gamma := \{ f(z) \mid z \in C \}$: the image of C under $w = f(z)$
Γ is closed contour, oriented, but not necessarily simple.

Let $w_0, w \in \Gamma$



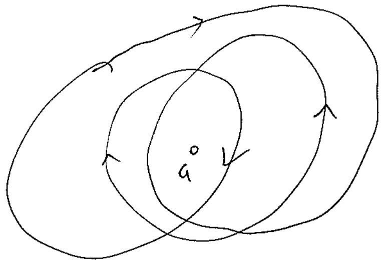
ϕ_0 is a value of $\arg w_0$
and let $\arg w$ vary continuously starting with ϕ_0.
When w returns to w_0, $\arg w$ is a value of $\arg w_0$, denoted by ϕ_1.

$$\Delta_C \arg f(z) := \phi_1 - \phi_0 \in \mathbb{Z} \cdot 2\pi i \Rightarrow \frac{1}{2\pi i} \Delta_C \arg f(z) = \tilde{\phi}_1 - \tilde{\phi}_0$$

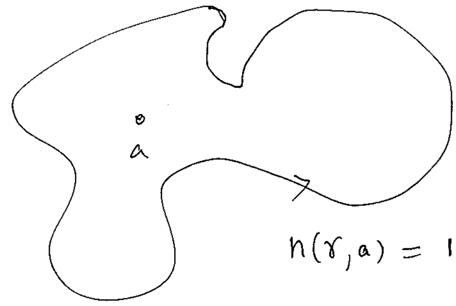
$\tilde{\phi}_1 = \phi_1 / 2\pi$

- $\frac{1}{2\pi i}\Delta_C \arg f(z)$: the winding number of Γ w.r.t. $w=0$...

- In general, $n(\gamma, a) := \frac{1}{2\pi i}\oint_\gamma \frac{dz}{z-a}$: the winding number of γ w.r.t. $w=a$.



$n(\gamma, a) = -3$.



$n(\gamma, a) = 1$

Theorem (Argument principle). Let C be a positively oriented simple closed contour and D_C be the domain bounded by C. Suppose that

1) f is meromorphic in D_C
2) f is analytic on C & $f(z) \neq 0 \ \forall z \in C$
3) $Z = \#$ zeros of f, $P = \#$ poles of f in D_C, counting multiplicities

Then
$$\frac{1}{2\pi i}\Delta_C \arg f(z) = Z - P = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz$$

Proof

- $C: \ z = z(t), \quad a \le t \le b, \quad z(a) = z(b)$
- $f[z(t)] = \rho(t)\, e^{i\phi(t)} \quad a \le t \le b$
- $f'(z(t)).z'(t) = \frac{d}{dt}[f(z(t))] = \rho'(t)\, e^{i\phi(t)} + i\rho(t)\, e^{i\phi(t)}.\phi'(t)$
- $\frac{f'(z(t)).z'(t)}{f(z(t))} = \frac{\rho'(t)}{\rho(t)} + i\phi'(t), \quad a \le t \le b$
- $\int_C \frac{f'(z)}{f(z)}\,dz = \int_a^b \frac{f'(z(t)).z'(t)}{f(z(t))}\,dt = \underbrace{\ln \rho(t)\Big|_a^b}_{=0} + \underbrace{i\phi(t)\Big|_a^b}_{=i(\phi(b)-\phi(a))}$
$= i\Delta_C \arg f(z)$.

$$\int_C \frac{f'(z)}{f(z)}\,dz = 2\pi i \sum_{j=1}^{n} \operatorname{Res}_{z_j} \frac{f'(z)}{f(z)}.$$

- if z_0 is a zero of order m_0,

$$f(z) = (z-z_0)^{m_0} g(z), \quad g(z_0) \neq 0$$

$$\frac{f'(z)}{f(z)} = \frac{m_0}{z-z_0} + \underbrace{\frac{g'(z)}{g(z)}}_{\text{holo.}}, \quad g(z) \neq 0 \ \forall z \in \text{nbd of } z_0.$$

$$\Rightarrow \operatorname{Res}_{z_0} \frac{f'(z)}{f(z)} = m_0 \geq 1$$

- if z_0 is a pole of order m_0

$$f(z) = \frac{g(z)}{(z-z_0)^{m_0}}$$

$$\frac{f'(z)}{f(z)} = -\frac{m_0}{z-z_0} + \underbrace{\frac{g'(z)}{g(z)}}_{\text{holo.}}$$

$$\Rightarrow \operatorname{Res}_{z_0} \frac{f'(z)}{f(z)} = -m_0.$$

Thus $\displaystyle\int_C \frac{f'(z)}{f(z)}\,dz = 2\pi i\,(Z - P)$

& $\displaystyle\frac{1}{2\pi} \Delta_C \arg f(z) = Z - P$ $\boxtimes$

<u>Ex.</u> 1) $\displaystyle\frac{1}{2\pi} \Delta_{|z|=1} \arg(z^2) = 2$, $\displaystyle\frac{1}{2\pi} \Delta_{\{|z|=1\}} \arg\left(\frac{1}{z^2}\right) = -2.$

2) $\displaystyle\frac{1}{2\pi} \Delta_{|z|=1} \arg(z^2+2) = 0.$

87. <u>Rouché's theorem</u>

Theorem (Rouché's theorem). Let C be a simple closed contour & D be a domain bounded by C. Suppose that

a) $f, g \in Hol(\overline{D})$

b) $|f(z)| > |g(z)| \quad \forall z \in C$

Then $f(z)$ and $f(z) + g(z)$ have the same number of zeros, counting multiplicities in D

<u>Proof</u>. + Suppose that the orientation of C is positive (counter-clockwise)

+ $|f(z)| > 0 \quad \forall z \in C$, $\quad |f(z) + g(z)| \geq |f(z)| - |g(z)| > 0 \quad \forall z \in C$

$\Rightarrow f, f+g$ have no zeros on C

+ By Argument principle,

$$Z_f = \frac{1}{2\pi} \Delta_C \arg f(z)$$

$$Z_{f+g} = \frac{1}{2\pi} \Delta_C \arg (f(z) + g(z))$$

$$= \frac{1}{2\pi} \Delta_C \arg \left[f(z) \left(1 + \frac{g(z)}{f(z)} \right) \right]$$

$$= \frac{1}{2\pi} \Delta_C \arg f(z) + \underbrace{\frac{1}{2\pi} \Delta_C \arg \left(1 + \frac{g(z)}{f(z)} \right)}_{= 0}$$

$\left|\frac{g}{f}\right| < 1$ on C

$$= \frac{1}{2\pi} \Delta_C \arg f(z) = Z_f \quad \boxed{\times}$$

~~Since $\left|\frac{g(z)}{f(z)}\right| < 1 \quad \forall z \in C \Rightarrow \frac{1}{2\pi} \Delta_C \arg \left(1 + \frac{g(z)}{f(z)}\right) = Z_{\frac{g}{f}+1}$~~ $= 0$

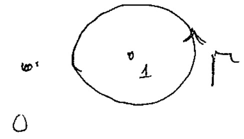
$\Downarrow$

~~$0 < 1 - \left|\frac{g(z)}{f(z)}\right| \leq \left|\frac{g(z)}{f(z)} + 1\right|$ has no zero~~

$$F(z) = 1 + \frac{g(z)}{f(z)} \Rightarrow |F(z) - 1| = \left|\frac{g(z)}{f(z)}\right| < 1 \text{ on } C$$

$\Rightarrow$ the image Γ of C under $F(z)$, $\Gamma \subset \{|w-1|<1\}$

$$\Rightarrow n(\Gamma, 0) = \frac{1}{2\pi}\Delta_C \arg\left(1+\frac{g(z)}{f(z)}\right) = 0.$$



Example

1) $z^4 + 3z^3 + 6$ has 3 zeros in $\{|z|<2\}$

$C = \{|z| = 2\}$

- $f(z) = 3z^3$ has 3 zeros in $\{|z|<2\}$
- $g(z) = z^4 + 6$, $\left|\frac{g(z)}{f(z)}\right| = \left|\frac{z^4+6}{3z^3}\right| \le \left.\frac{|z|^4+6}{3|z|^3}\right|_{|z|=2} = \frac{16+6}{3.8}$

$$\le \frac{22}{24} < 1 \quad \forall |z| = 2.$$

By Rouche's theorem, $f(z)+g(z)$ has 3 zeros in $\{|z|\le 2\}$

2) Fundamental theorem of algebra.

$P_n(z) = a_n z^n + \dots + a_1 z + a_0$, $a_n \ne 0$

has n solutions.

Proof $f(z) = a_n z^n$ has n solutions in $\{|z|<R\}$

$g(z) = a_{n-1}z^{n-1} + \dots + a_1 z + a_0$

$|f(z)|_{|z|=R} = |a_n| R^n$, $R > 1$

$$\frac{|g(z)|}{|f(z)|} \le \frac{|a_{n-1}|R^{n-1} + \dots + |a_1|R + |a_0|}{|a_n| R^n} = \frac{1}{R}\cdot\frac{|a_{n-1}| + \dots + |a_0|}{|a_n|} < 1$$

if we choose $R > \max\left\{\frac{|a_{n-1}| + \dots + |a_0|}{|a_n|}, 1\right\}$

$\Rightarrow$ By Rouche's theorem, $f+g = P_n(z)$ has n solutions in $\mathbb{C}$.

88. Inverse Laplace transforms

$f : [0, +\infty) \longrightarrow \mathbb{R}$

$$F(s) = \int_0^{+\infty} e^{-st} f(t)\, dt$$: the Laplace transform of $f(t)$

$f(t)$	$F(s)$
1	$\frac{1}{s}$, $\operatorname{Re} s > 0$
e^{at}	$\frac{1}{s-a}$, $\operatorname{Re} s > a$
t^n	$\frac{n!}{s^{n+1}}$, $\operatorname{Re} s > 0$
t^p, $p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}$ $\operatorname{Re} s > 0$
$\sin at$	$\frac{a}{s^2+a^2}$, $\operatorname{Re} s > 0$
$\cos at$	$\frac{s}{s^2+a^2}$ $\operatorname{Re} s > 0$.

$$f(t) := \frac{1}{2\pi i} \lim_{R \to \infty} \int_{\gamma - iR}^{\gamma + iR} e^{st} F(s)\, ds \qquad (t > 0)$$

$$f(t) = \frac{1}{2\pi i} \text{ P.V.} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} F(s)\, ds \qquad (t > 0)$$

: the inverse Laplace transform of $F(s)$,

where F is holomorphic in $\mathbb{C}$ except $\{s_1, \dots, s_N\}$

$\{s_1, \dots, s_N\}$ are isolated singular points of F.

$\gamma >> 0$ s.t. the singularities of F all lie to the left of the segment $\{\operatorname{Re} z = \gamma\}$.

(i.e. $\operatorname{Re} s_j < \gamma$) 20

$L_R = [\gamma - iR, \gamma + iR]$

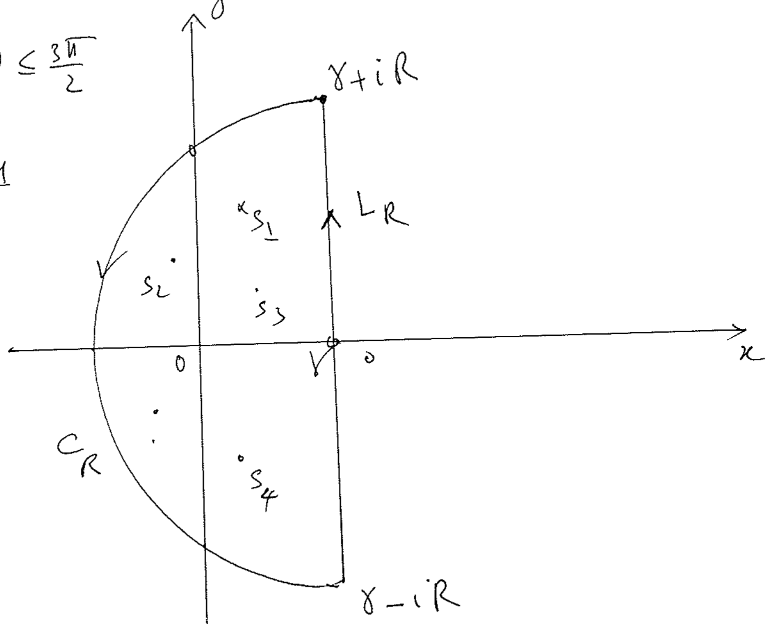
$C_R = \{|z-\gamma| = R,\ \operatorname{Re} z \le \gamma\}$

$z = \gamma + Re^{i\theta},\ \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$

We can choose $R \gg 1$ & $\gamma \gg 1$ s.t

$s_1, .., s_N \subset D_R$

D_R is the domain bdd by $C_R \cup L_R$.



By Cauchy's residue theorem, we obtain that

$$\int_{L_R} e^{st}F(s)\,ds + \int_{C_R} e^{st}F(s)\,ds = 2\pi i \sum_{j=1}^{N} \operatorname*{Res}_{s=s_j}[e^{st}F(s)]$$

$$\Rightarrow \int_{L_R} e^{it}F(s)\,ds = 2\pi i \sum_{j=1}^{N} \operatorname*{Res}_{s=s_j}[e^{st}F(s)] - \underbrace{\int_{C_R} e^{st}F(s)\,ds}_{\downarrow R\to+\infty \atop 0}$$

$$\left|\int_{C_R} e^{st}F(s)\,ds\right| = \left|\int_{\pi/2}^{3\pi/2} e^{t(\gamma+Re^{i\theta})} F(\gamma+Re^{i\theta})\, Rie^{i\theta}\, d\theta\right|$$

$$\le \int_{\pi/2}^{3\pi/2} e^{t\gamma}\cdot e^{tR\cos\theta} M_R \cdot R\, d\theta, \quad M_R = \sup_{s\in C_R}|F(s)|.$$

$$= M_R\cdot R\cdot e^{t\gamma}\cdot \int_{\pi/2}^{3\pi/2} e^{tR\cos\theta}\, d\theta$$

$\phi = \theta - \pi/2$

$$0 \le \left|\int_{C_R} e^{st}F(s)\,ds\right| \le M_R\cdot R\, e^{t\gamma}\int_0^{\pi} e^{-tR\sin\phi}\, d\phi \le M_R\cdot \cancel{R}\cdot e^{t\gamma}\cdot \frac{\pi}{\cancel{R}t}$$

$\sin\phi \ge \frac{\pi}{2}\theta$

$\to 0$ $\to 0$ if $M_R \to 0$ as $R \to +\infty$.

$$\Rightarrow \lim_{R\to+\infty} \int_{C_R} e^{st}F(s)\,ds$$

$$\Rightarrow f(t) = \sum_{j=1}^{N} \operatorname*{Res}_{s=s_j}\left[e^{st}F(s)\right]$$

In general

$$f(t) = \sum_{j=1}^{\infty} \operatorname*{Res}_{s=s_j}\left[e^{st}F(s)\right].$$

Condition: $M_R \longrightarrow 0$ as $R \to +\infty$,

89. Examples

Remark F has a pole of order $m \geq 1$ at s_0

$$F(s) \overset{\text{Laurent series}}{=} \frac{b_m}{(s-s_0)^m} + \frac{b_{m-1}}{(s-s_0)^{m-1}} + \dots + \frac{b_1}{s-s_0} + a_0 + \dots \quad (b_m \neq 0)$$

$$e^{st}F(s) = b_m \frac{e^{st}}{(s-s_0)^m} + \dots + b_1 \frac{e^{st}}{s-s_0} + a_0 e^{st} + a_1 e^{st}(s-s_0) + \dots$$

- $$\operatorname*{Res}_{s=s_0}\left[e^{st}F(s)\right] = e^{s_0 t}\left[b_1 + \frac{b_2}{1!}t + \dots + \frac{b_m}{(m-1)!}\cdot t^{m-1}\right].$$

$$s_0 = \alpha + i\beta$$

- $$\operatorname*{Res}_{s=s_0}\left[e^{st}F(s)\right] + \operatorname*{Res}_{s=\bar{s}_0}\left[e^{st}F(s)\right] = 2e^{\alpha t}\,\mathrm{Re}\left\{e^{i\beta t}\left(b_1 + b_2\frac{t}{1!} + \dots + b_m\frac{t^{m-1}}{(m-1)!}\right)\right\}$$

$$M_R = \max_{s\in C_R} F(s) \lesssim \frac{1}{R} \to 0 \text{ as } R\to+\infty$$

Example 1.

$$F(s) = \frac{s^2 - a^2}{(s + a^2)^2} \quad (a > 0) \longrightarrow f(t) = ?$$

$$s_0 = ai, \ \bar{s}_0 = -ai, \quad F = \frac{b_1}{s-ai} + \frac{b_2}{(s-ai)^2} + a_0 + a_1(s-ai) + \cdots$$

$$\operatorname{Res}_{s=ai}\left[e^{st}F(s)\right] + \operatorname{Res}_{s=-ai}\left[e^{st}F(s)\right] = 2\operatorname{Re}\left(e^{iat}(b_1 + b_2 t)\right)$$
$$= 2t\tfrac{1}{2}\cos at = t\cos at.$$

$$F(s) = \frac{s^2-a^2}{(s-ai)^2(s+ai)^2} = \frac{\phi(s)}{(s-ai)^2} = \frac{\phi(ai) + \phi'(ai)(s-ai) + \cdots}{(s-ai)^2}$$

$$= \frac{\phi(ai)}{(s-ai)^2} + \frac{\phi'(ai)}{s-ai}, \quad b_1 = \phi'(ai), \ b_2 = \phi(ai)$$

$$\phi = \frac{s^2-a^2}{(s+ai)^2}, \quad \phi(ai) = \frac{(ai)^2 - a^2}{(2ai)^2} = \frac{-2a^2}{-4a^2} = \frac{1}{2}$$

$$\phi'(ai) = \left.\frac{2s(s+ai) - 2(s^2-a^2)}{(s+ai)^2}\right|_{s=ai} = 0$$

$$b_1 = 0, \ b_2 = 1/2$$

Thus $\quad f(t) = 2\operatorname{Re}\left(e^{iat}(b_1 + b_2 t)\right)$

$$= 2 \cdot \tfrac{1}{2} \cdot t\cos at = t\cos at.$$

Check: $\quad M_R = \max_{s \in C_R} |F(s)|$

$$C_R = \left\{ z = \gamma + Re^{i\theta} \ \middle| \ \tfrac{\pi}{2} \le \theta \le \tfrac{3\pi}{2} \right\}; \ \gamma > 0$$

$$|F(s)| = \left|\frac{s^2 - a^2}{(s^2+a^2)^2}\right| \le \frac{|s|^2 + a^2}{(|s|^2 - a^2)^2}, \quad R-\gamma < |s| \le R + \gamma$$

$$\le \frac{(|\gamma| + R)^2 + a^2}{\left((R-\gamma)^2 - a^2\right)^2} \longrightarrow 0 \text{ as } R \to +\infty \quad \forall s \in C_R.$$

$\Rightarrow \quad M_R \to 0$ as $R \to +\infty$

Example 2.

$$F(s) = \frac{1}{s^2} - \frac{1}{s\sinh s}.$$

$$\sinh s = \frac{e^s - e^{-s}}{2} = 0 \Leftrightarrow e^{2s} = 1$$

$$\Leftrightarrow 2s = 2n\pi i$$

$$\Leftrightarrow s = n\pi i, \quad n \in \mathbb{Z}$$

$$\sinh(s) = \frac{1 + s + \frac{s^2}{2!} - \left(1 - s + \frac{s^2}{2!} + \dots\right)}{2}$$

$$= \frac{2s + \frac{2s^3}{6} + \dots}{2} = s + \frac{s^3}{6} + \dots$$

$$F(s) = \frac{1}{s^2} - \frac{1}{s^2} \cdot \frac{1}{1 + \frac{s^2}{6} + \dots}$$

$$= \frac{1}{s^2} - \frac{1}{s^2} \cdot \left(1 - \frac{s^2}{6} + \dots\right)$$

$\Rightarrow$ $s = 0$ is a removable singular pt of F

$$\sinh'(s) = \frac{e^s + e^{-s}}{2} = \frac{e^{2n\pi i} + 1}{2e^{n\pi i}} = \frac{1}{e^{n\pi i}}$$

$s = n\pi i$ is a pole of order 1 (simple pole)

$$\operatorname*{Res}_{s=n\pi i} F(s) = \frac{-\frac{1}{s}}{(\sinh s)'} = -\frac{1}{n\pi i\, e^{n\pi i}} = b_1$$

$$\operatorname*{Res}_{s=n\pi i}\left[e^{st}F(s)\right] + \operatorname*{Res}_{s=-n\pi i}\left[e^{st}F(s)\right] = 2\,\mathrm{Re}\left[e^{in\pi t}\left(-\frac{1}{n\pi i\, e^{n\pi i}}\right)\right]$$

$n = 1, 2 \dots$

$$= -2\,\mathrm{Re}\, \frac{1}{n\pi i} \cdot e^{in\pi(t-1)}.$$

$$= 2\,\mathrm{Re}\left[i\, \frac{e^{in\pi(t-1)}}{n\pi}\right] = -\frac{2}{n\pi}\sin(n\pi(t-1))$$

$$= \frac{2}{\pi} \cdot \frac{(-1)^{n+1}}{n} \sin(n\pi t)$$

$$f(t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n\pi t), \quad t \in \mathbb{R}$$

Check.

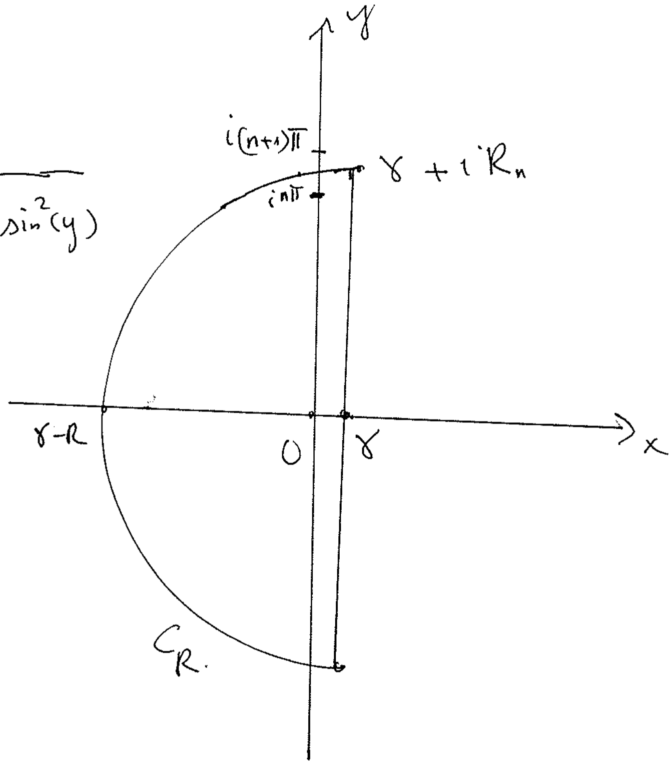
$0 < \gamma << 1$

$$R_n = \frac{2n+1}{2} \pi$$

$$|\sinh(s)| = \sqrt{\sinh^2(x) + \sin^2(y)}$$

$$\geq \varepsilon_0 > 0$$

$\forall \; s \in C_R$



$\Rightarrow$

$$M_{R_n} := \max_{s \in C_R} |F(s)|$$

$$\leq \frac{1}{R_n^2} + \frac{1}{R_n \cdot \varepsilon_0} \longrightarrow 0 \quad \text{as } R_n \to +\infty$$

Chapter 8. Mapping by elementary functions

90. Linear transformations

$$w = az + b \; : \; \mathbb{C} \longrightarrow \mathbb{C} \quad (a \neq 0)$$
$$= f(z) = az + b$$

$\cdot\ w_1 = z + b =: f_1(z) : \mathbb{C} \longrightarrow \mathbb{C}$: translation

$z \longmapsto z + b, \quad b = (b_1, b_2)$

$\cdot\ w_\cdot = az = f_2(z) : \mathbb{C} \longrightarrow \mathbb{C}$:

$z \longmapsto re^{i\theta} \cdot z$

f_2 expands or contracts by factor r and rotates the angle θ

$\Rightarrow f = f_1 \circ f_2 : \mathbb{C} \longrightarrow \mathbb{C}$

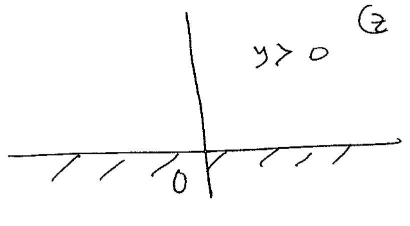
Ex $\quad w = (1+i)z \quad (1)$

Find the image of $\{y > 0\}$ by (1)

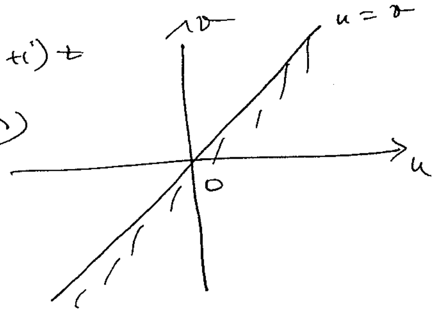
$$w = u + iv = (1+i)(x+iy)$$
$$\Rightarrow \quad = x - y + i(x+y)$$

$\begin{cases} u = x - y \\ v = x - y \end{cases} \Rightarrow u - v = -2y \Rightarrow v - u = 2y > 0$

$\Rightarrow \boxed{v > u}$



$w = (1+i)z$



91. & 92

The transformation $w = \frac{1}{z}$

$$w = \frac{1}{z} : \mathbb{C}\setminus\{0\} \longrightarrow \mathbb{C}$$
$$z \longmapsto \frac{1}{z}.$$
$$= \frac{\bar{z}}{|z|^2}$$

$z.$

$$0 \longmapsto \infty$$
$$\infty \longmapsto 0$$

$$w = T(z) : \overline{\mathbb{C}} \xrightarrow{1-1} \overline{\mathbb{C}} : \text{holomorphic}$$

$$w = \frac{\bar{z}}{|z|^2} = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2} = u + iv$$

$$x = \frac{u}{u^2+v^2}$$

$$\Rightarrow \begin{cases} u = \dfrac{x}{x^2+y^2} \\ v = -\dfrac{y}{x^2+y^2} \end{cases} \qquad y = -\frac{v}{u^2+v^2}$$

<u>Theorem</u>. the mapping $w = \frac{1}{z}$ transforms circles and lines into circles and lines.

Proof

Circles or lines : $A(x^2+y^2) + Bx + Cy + D = 0$ (1)
$A, B, C, D \in \mathbb{R}$

<u>Condition</u> $B^2 + C^2 > 4AD$

$$(1) \Leftrightarrow \left(x + \frac{B}{2A}\right)^2 + \left(y + \frac{C}{2A}\right)^2 = \left(\frac{\sqrt{B^2+C^2-4AD}}{2A}\right)^2$$

$$\Downarrow \quad x =$$

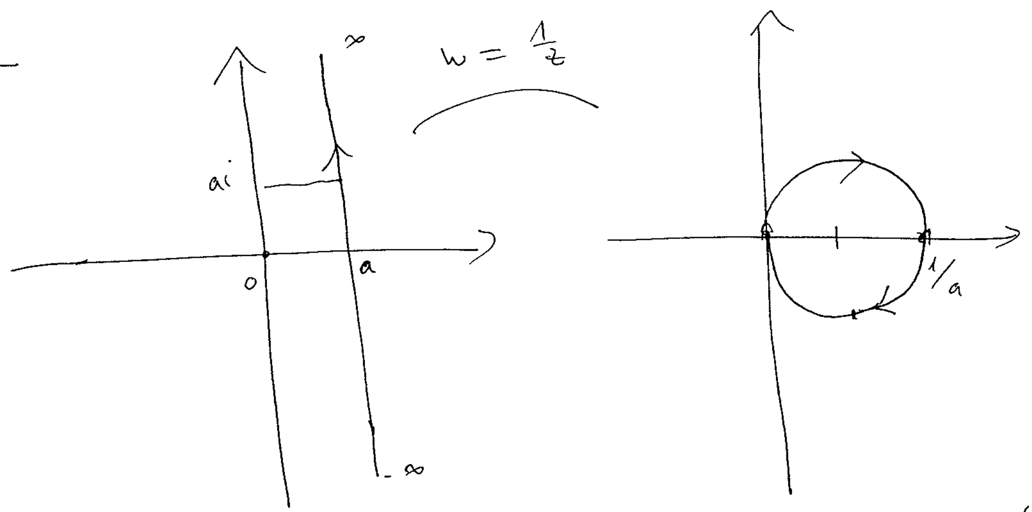
$$\frac{A(u^2+v^2)}{(u^2+v^2)^2} + \frac{B.u \cancel{+} Cv}{u^2+v^2} + D = 0$$

$$\Leftrightarrow \quad D(u^2+v^2) + Bu - Cv + A = 0$$

$\uparrow$

circles or lines. $\boxtimes$.

Example



$$\frac{1}{a(1+i)} = \frac{1-i}{2a}.$$

$$\{x=0\} \longrightarrow \{v=0\}$$

$$\{y=0\} \longrightarrow \{u=0\}$$

$$\{x=a>0\} \longmapsto \{(u-\tfrac{1}{2a})^2+v^2 = (\tfrac{1}{2a})^2\}$$

$$\ldots$$

$$\{x>a\} \longmapsto \{(u-\tfrac{1}{2a})^2+v^2 < (\tfrac{1}{2a})^2\}.$$

93. Linear fractional transformations

$$w = \frac{az+b}{cz+d} \qquad (ad-bc \neq 0)$$

$$= \frac{a}{c} + \frac{bc-ad}{c} \cdot \frac{1}{cz+d} \qquad (c \neq 0)$$

$$Z = cz+d \; : \; \overline{\mathbb{C}} \to \overline{\mathbb{C}}$$

$$W = \frac{1}{Z} \; : \; \overline{\mathbb{C}} \to \overline{\mathbb{C}}$$

$$w = \frac{a}{c} + \frac{bc-ad}{c} \cdot W \; : \; \overline{\mathbb{C}} \to \overline{\mathbb{C}}$$

Theorem. the mapping $w = \dfrac{az+b}{cz+d}$ ~~ma~~ transforms circles and lines into circles and lines.

$$w = \frac{az+b}{cz+d} = T(z) \; : \; \overline{\mathbb{C}} \xrightarrow{1-1} \overline{\mathbb{C}}$$

$$-\frac{d}{d} \longmapsto \infty$$

$$\infty \longmapsto \frac{a}{c}$$

$$z = \frac{-dw+b}{cw-a} = T^{-1}(w) \; : \; \overline{\mathbb{C}} \to \overline{\mathbb{C}}$$

$$\infty \longmapsto -\frac{d}{c}$$

$$\frac{a}{c} \longmapsto \infty$$

Example

Find the linear transformation that maps $2 \mapsto 1$, $i \mapsto i$, $-2 \mapsto -1$

$$w = \frac{az+b}{cz+d}$$

$$\begin{cases} 1 = \dfrac{2a+b}{2c+d} \\[2mm] i = \dfrac{ai+b}{ci+d} \\[2mm] -1 = \dfrac{-2a+b}{-2c+d} \end{cases} \longrightarrow \begin{cases} 2c+d = 2a+b \\ -c+di = ai+b \\ 2c-d = -2a+b \end{cases}$$

$$\longrightarrow \begin{cases} b = 2c \\ d = 2a \\ -c+ai = 2c \end{cases} \Rightarrow \begin{cases} b = 2c \\ a = -3ic \\ d = -6ic \end{cases}$$

$$\Rightarrow \quad w = \frac{-3iz+2}{z-6i} = \frac{3z+2i}{iz+6}.$$

94. An implicit form

$z_1 \mapsto w_1$

$z_2 \mapsto w_2$

$z_3 \mapsto w_3$

$$\Rightarrow \quad \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

Ex. $2 \mapsto 1$

$i \mapsto i$

$-2 \mapsto -1$

$$\Rightarrow \quad \frac{(w-1)(i+1)}{(w+1)(i-1)} = \frac{(z-2)(i+2)}{(z+2)(i-2)}$$

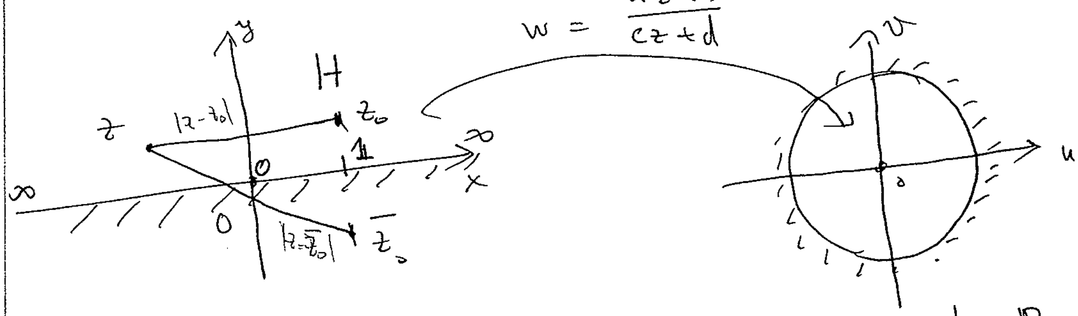
$$\Rightarrow \quad \frac{w-1}{w+1} \cdot (-i) = \frac{z-2}{z+2}\left(-\frac{4i+3}{5}\right)$$

$$\Rightarrow \quad w = \frac{3z+2i}{iz+6} \quad \boxtimes$$

95. Mappings of the upper half plane

- $\mathbb{H} = \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \}$, $\mathbb{D} = \{ w \in \mathbb{C} \mid |w| < 1 \}$

$$w = \frac{az+b}{cz+d}$$



Question: Find $w = \dfrac{az+b}{cz+d}$, maps $\mathbb{H}$ onto $\mathbb{D}$

Answer: $w = e^{i\alpha} \cdot \dfrac{z - z_0}{z - \bar{z}_0}$, $\operatorname{Im} z_0 > 0$

Proof

"$\Leftarrow$" • $\forall z \in \mathbb{H} \Rightarrow |z - z_0| < |z - \bar{z}_0|$

$$\Rightarrow |w| = |e^{i\alpha}| \left| \frac{z - z_0}{z - \bar{z}_0} \right| < 1$$

• $\forall z \in H^- \Rightarrow |z - z_0| > |z - \bar{z}_0|$

$$\Rightarrow |w| = |e^{i\alpha}| \left| \frac{z - z_0}{z - \bar{z}_0} \right| > 1$$

• $\forall z \in \partial\mathbb{H} \Rightarrow |z - z_0| = |z - \bar{z}_0| \Rightarrow |w(z)| = 1$.

Thus $w = e^{i\alpha} \dfrac{z - z_0}{z - \bar{z}_0}$ maps $\mathbb{H}$ onto $\mathbb{D}$

"$\Rightarrow$" $w = \dfrac{az+b}{cz+d} = f(z)$, $a, b, c, d = ?$

$f : \mathbb{H} \xrightarrow{\text{onto}} \mathbb{D}$, $f(\partial\mathbb{H}) = \partial\mathbb{D}$

$0, 1, \infty \longmapsto f(0), f(1), f(\infty) \in \partial\mathbb{D}$

$|f(0)| = 1 \iff \left|\dfrac{b}{d}\right| = 1 \Rightarrow |b| = |d| \neq 0$

$|f(1)| = 1 \iff \left|\dfrac{a+b}{c+d}\right| = 1 \Rightarrow \left|1 + \dfrac{b}{a}\right| = \left|1 + \dfrac{d}{c}\right| \Rightarrow$

$|f(\infty)| = 1 \iff \left|\dfrac{a}{c}\right| = 1 \Rightarrow |a| = |c|$

$$w = \frac{a}{c} \cdot \frac{z + b/a}{z + d/c} = e^{i\alpha} \cdot \frac{z - z_0}{z - z_1}$$

Since $|f(1)| = 1 \Rightarrow \dfrac{|1 - z_0|}{|1 - z_1|} = 1$, $|z_0| = |z_1|$

$$\Leftrightarrow \begin{cases} |1 - z_0| = |1 - z_1| \\ |z_0| = |z_1| \end{cases} \Rightarrow \left[\begin{array}{l} z_0 = z_1 \\ z_1 = \bar{z}_0 \end{array}\right.$$

$$\Rightarrow \quad w = e^{i\alpha} \frac{z - z_0}{z - \bar{z}_0} .$$

Example

1) $w = \dfrac{i - z}{i + z} = e^{\pi i} \dfrac{z - i}{z + i}$: $\mathbb{H} \xrightarrow{1-1} \mathbb{D}$

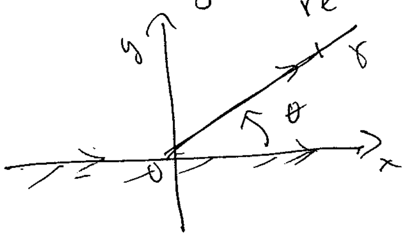
2) $w = \dfrac{z - 1}{z + 1}$: $\mathbb{H} \longrightarrow \mathbb{H}$

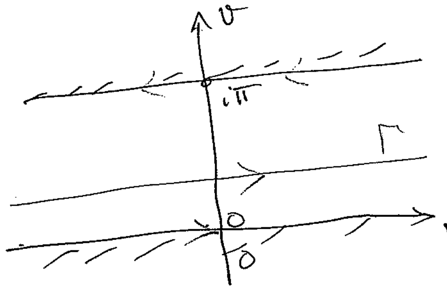
$$w = \frac{z-1}{z+1} \Rightarrow wz + w = z - 1$$

$$\Rightarrow z = \frac{w+1}{1-w} = \frac{(w+1)(1-\bar{w})}{|1-w|^2} = \frac{1 - |w|^2 + w - \bar{w}}{|1-w|^2}$$

$\operatorname{Im} z > 0 \Leftrightarrow 2 \operatorname{Im} w > 0 \Leftrightarrow \operatorname{Im} w > 0$ $\boxtimes$.

3) $w = \operatorname{Log} z$





$\Gamma = \operatorname{Log} \gamma$.

$$\operatorname{Log}(re^{i\theta}) = \ln|r| + i\theta$$

$$0 \leq \theta \leq \pi$$

Remark. $\operatorname{Aut}(\mathbb{D}) = \{ f : \mathbb{D} \xrightarrow{1-1} \mathbb{D} : \text{biholomorphic} \}$

$$= \left\{ z \mapsto e^{i\varphi} \frac{z - \alpha}{1 - \bar{\alpha} z} \;\middle|\; \varphi \in \mathbb{R}, |\alpha| < 1 \right\}$$

Elementary mappings (§96 ÷ §100)

- Joukowsky transform $w = \frac{1}{2}\left(z + \frac{1}{z}\right)$
- $w = \cos z$
- $w = \sin z$
- $w = z^2$, $w = z^n$
- $w = \sqrt{z}$, $w = \sqrt[n]{z}$
- Square roots of polynomials
  - $w = \sqrt{(z-a)(z-b)}$
  - $w = \sqrt{\frac{z-a}{z-b}}$
- Riemann surfaces
- Surfaces for related functions

1. Joukowsky transform

$$w = \frac{1}{2}\left(z + \frac{1}{z}\right) : \mathbb{C}\setminus\{0\} \longrightarrow \mathbb{C}$$

$$w_1 = \frac{1}{2}\left(z_1 + \frac{1}{z_1}\right)$$

$$w_2 = \frac{1}{2}\left(z_2 + \frac{1}{z_2}\right)$$

$$w_1 - w_2 = \frac{1}{2}(z_1 - z_2)\left(1 - \frac{1}{z_1 z_2}\right)$$

$\Rightarrow$ $w = \frac{1}{2}\left(z + \frac{1}{z}\right)$ is one-to-one on D iff

$\nexists\, z_1, z_2 \in D$ s.t. $z_1 \cdot z_2 = 1$.

<u>Example</u> $w = \frac{1}{2}\left(z + \frac{1}{z}\right)$ is $1-1$ on $\{|z|<1\}$
& $\{|z|>1\}$

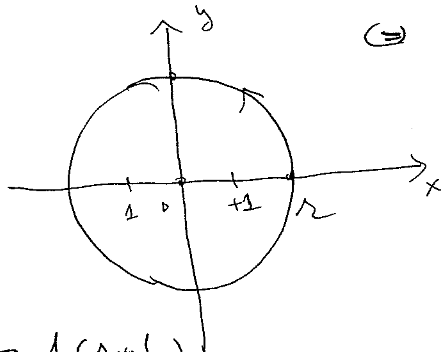
- $w = \frac{1}{2}\left(z + \frac{1}{z}\right)$ maps $\{|z|=1\}$ onto $[-1, 1]$
 maps $\{|z|=r\}$ onto ellipse, foci at ± 1

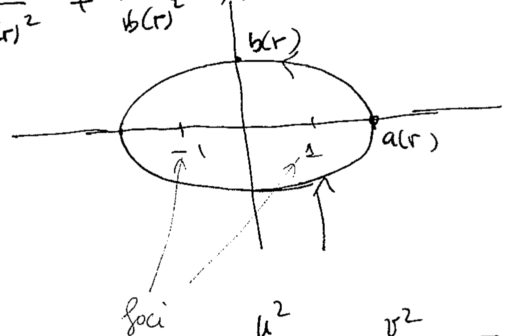
$$z = re^{i\theta}, \quad w = \frac{1}{2}\left(re^{i\theta} + \frac{1}{r}e^{-i\theta}\right) = u + iv$$

$$= \frac{1}{2}\left(r + \frac{1}{r}\right)\cos\theta + i\frac{1}{2}\left(r - \frac{1}{r}\right)\sin\theta$$

$$\Rightarrow \begin{cases} u = \frac{1}{2}\left(r + \frac{1}{r}\right)\cos\theta \\ v = \frac{1}{2}\left(r - \frac{1}{r}\right)\sin\theta \end{cases} \Rightarrow \frac{u^2}{\frac{1}{4}\left(r+\frac{1}{r}\right)^2} + \frac{v^2}{\frac{1}{4}\left(r-\frac{1}{r}\right)^2} = 1$$

$$\Leftrightarrow \frac{u^2}{a(r)^2} + \frac{v^2}{b(r)^2} = 1$$

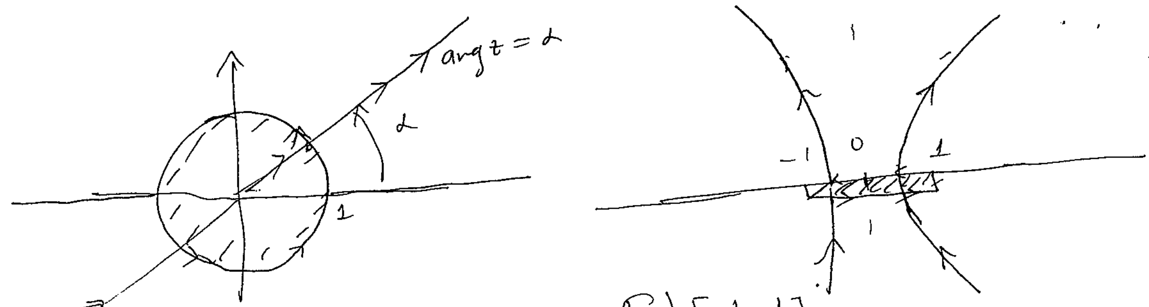




$a(r) = \frac{1}{2}\left(r + \frac{1}{r}\right)$

$b(r) = \frac{1}{2}\left|r - \frac{1}{r}\right|$

$$\frac{u^2}{a(r)^2} + \frac{v^2}{b(r)^2} = 1$$



$$\{|z|>1\} \longmapsto \mathbb{C}\setminus[-1,1]$$
$$\{|z|<1\} \longmapsto \mathbb{C}\setminus[-1,1].$$

- $w = \frac{1}{2}(z+\frac{1}{z})$ maps $\{\arg z = \alpha\} \longrightarrow$ hyperbola

$$z = re^{i\alpha}$$

$$\begin{cases} u = \frac{1}{2}(r+\frac{1}{r})\cos\alpha \\ v = \frac{1}{2}(r-\frac{1}{r})\sin\alpha \end{cases} \Rightarrow \frac{u^2}{\cos^2\alpha} - \frac{v^2}{\sin^2\alpha} = 1$$

2. $$w = \cos z = \frac{e^{iz}+e^{-iz}}{2} = \frac{1}{2}\left(e^{iz} + \frac{1}{e^{iz}}\right)$$

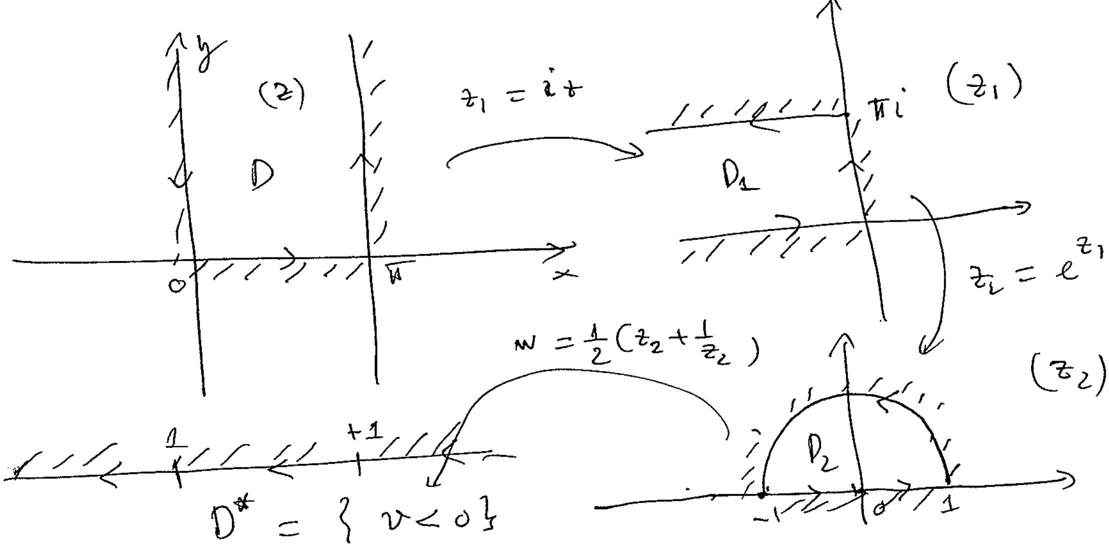
$$z_1 = iz$$

$$z_2 = e^{z_1}$$

$$z_3 = \frac{1}{2}\left(z_2 + \frac{1}{z_2}\right)$$

$$w = \cos z = z_3 \circ z_2 \circ z_1 .$$

Example. Find image of $D = \{z \in \mathbb{C} \mid y>0, 0 \le x \le \pi\}$ under $w = \cos z$.



3. $w = \sin z = \cos\left(z - \frac{\pi}{2}\right) = \dfrac{e^{iz} - e^{-iz}}{2i}$

$$= \frac{1}{2}\left(\frac{e^{iz}}{i} + \frac{i}{e^{iz}}\right) = \frac{1}{2}\left(-ie^{iz} + \frac{1}{-ie^{iz}}\right)$$

$$= z_4 \circ z_3 \circ z_2 \circ z_1$$

$$z_1 = iz$$

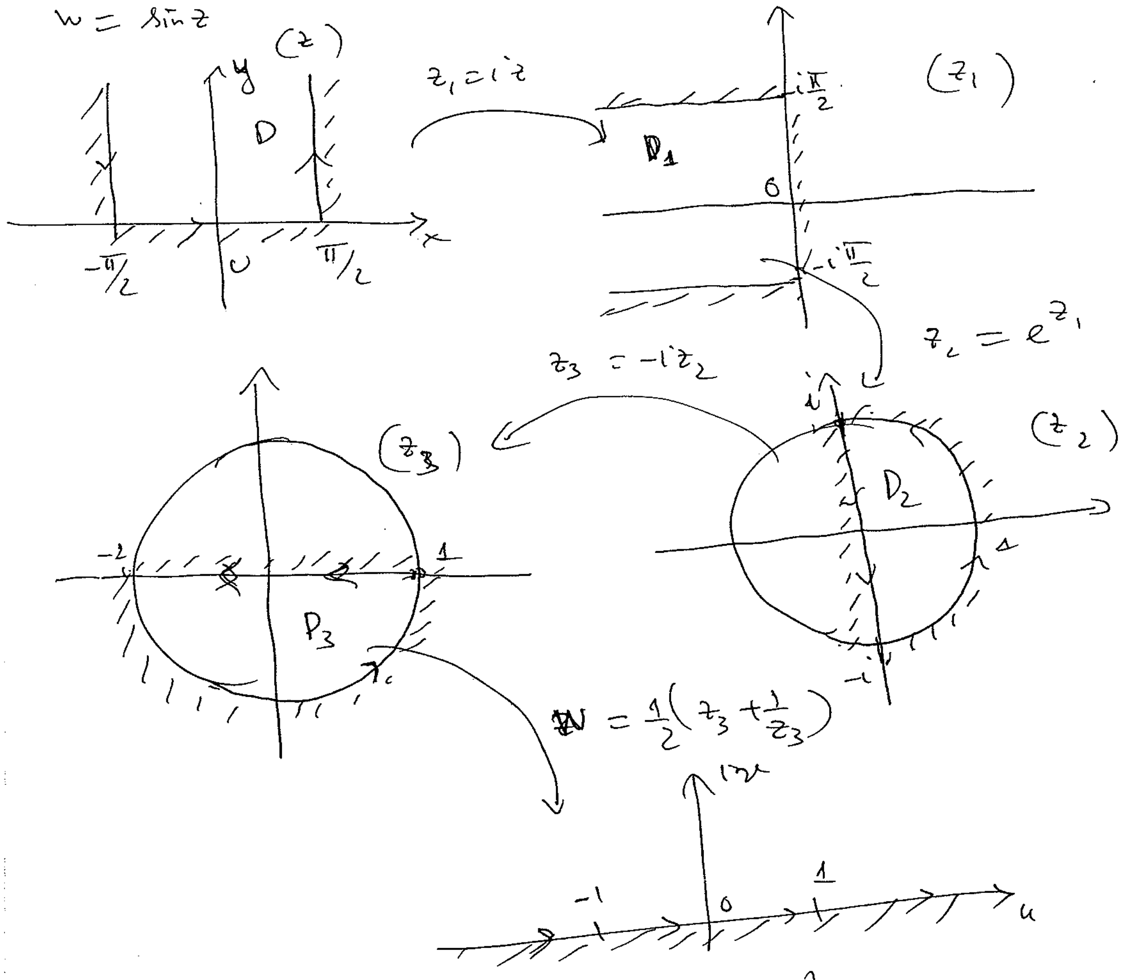
$$z_2 = e^{z_1}$$

$$z_3 = \frac{z_2}{i} = -iz_2 = e^{-i\frac{\pi}{2}} \cdot z_2$$

$$z_4 = \frac{1}{2}\left(z_3 + \frac{1}{z_3}\right)$$

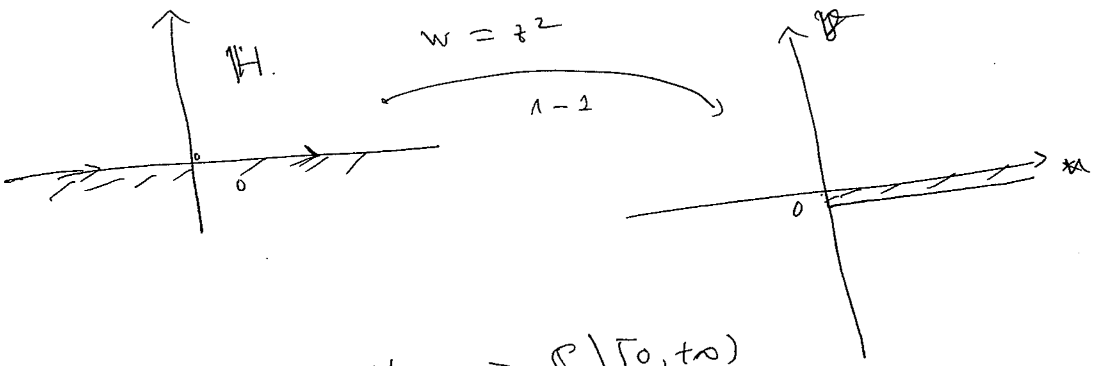
Example

Find $D^* =$ the image of $D = \{ y > 0, -\frac{\pi}{2} < x < \frac{\pi}{2} \}$ under $w = \sin z$



Answer: $D^* = \mathbb{H} = \{ v > 0 \}$.

4. $w = z^2 = x^2 - y^2 + 2xy\,i$, $u = x^2 - y^2$
$v = 2xy$



$w = z^2$ maps $\mathbb{H} \longrightarrow \mathbb{C} \setminus [0, +\infty)$

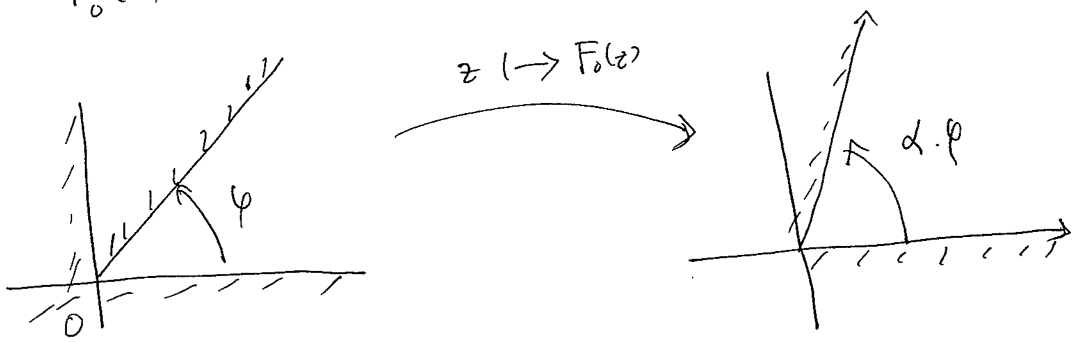
$w = \sqrt{z} = \sqrt{r}\, e^{i\frac{\theta}{2}} : \mathbb{C} \setminus [0, +\infty) \longmapsto \mathbb{H}$

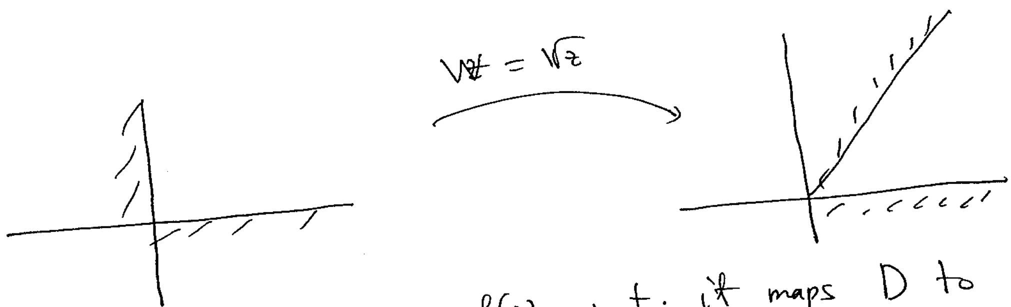
$0 \le \theta \le 2\pi$

$w = -\sqrt{z} = -\sqrt{r}\, e^{i\frac{\theta}{2}} : \mathbb{C} \setminus [0, +\infty) \longmapsto -\mathbb{H} = \mathbb{H}^- = \{ v < 0 \}.$

5 $w = z^\alpha = e^{\alpha \log z} = e^{\alpha (\ln|z| + i\theta_0 + 2k\pi i)}$

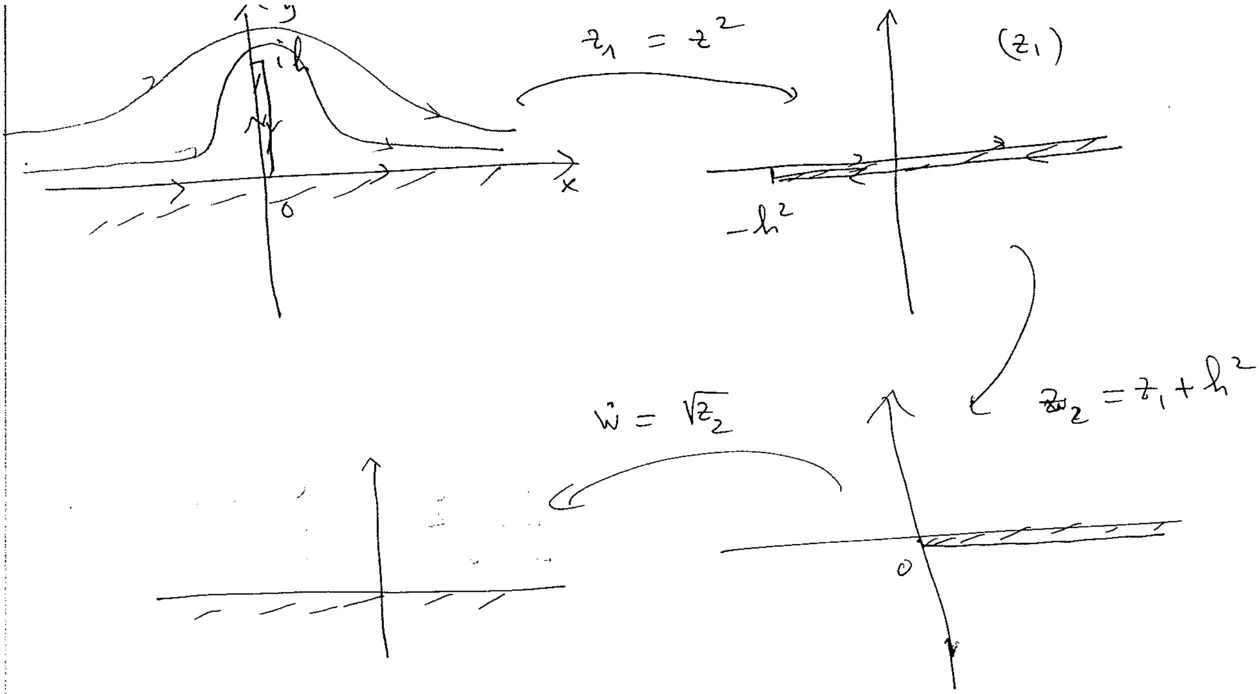
$\underline{k = 0}$ $F_0(z) = z^\alpha = e^{\alpha(\ln|z| + i\theta_0)}$, $0 \le \theta_0 \le 2\pi$





$\underline{\text{Example}}$ Find a map $w = f(z)$ s.t. it maps D to $\mathbb{H}$

$D = \{ y > 0 \} \setminus [0, ih]$ $h > 0$

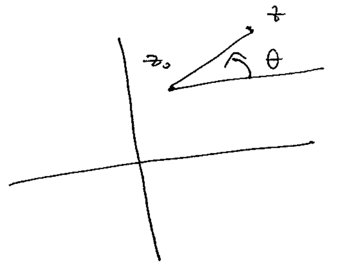


Answer, $w = \sqrt{z^2 + h^2}$

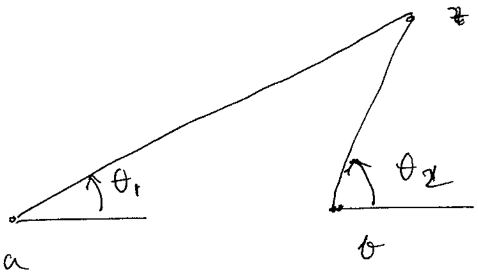
6. Square roots of polynomial

a) $w = \sqrt{z - z_0} = \sqrt{r}\, e^{i\frac{\theta + 2k\pi}{2}}$, $k = 0, 1$

$z - z_0 = r e^{i\theta}$



b) $w = \sqrt{(z-a)(z-b)}$



$z - a = r_1 e^{i\theta_1}$

$z - b = r_2 e^{i\theta_2}$

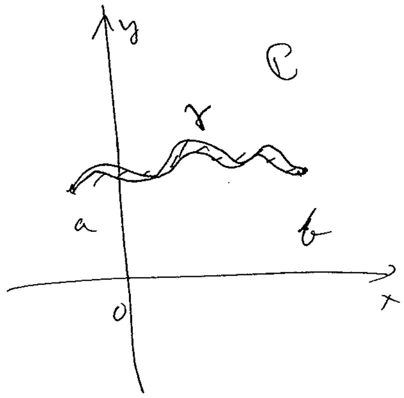
$$w = \sqrt{r_1 . r_2}\; e^{i\frac{\theta_1 + \theta_2}{2}}$$

$$= \sqrt{r_1 . r_2}\; e^{i\left(\frac{\theta_1^0 + \theta_2^0}{2} + \frac{2n_1\pi + n_2\pi}{2}\right)}$$

So $w = \sqrt{(z-a)(z-b)}$

has a branch defined on $\mathbb{C} \setminus \gamma$

γ : a curve connecting a and b



or $\mathbb{C} \setminus \gamma_a \cup \gamma_b$

γ_a : a curve connecting a and ∞

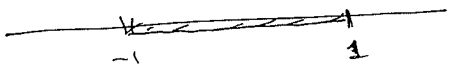
γ_b : ———————— b —— ∞

Example.

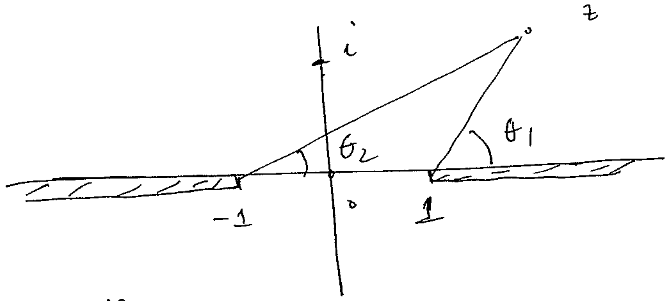
$$w = \underbrace{\sqrt{z^2-1}}_{= f(z)} = \sqrt{(z-1)(z+1)} \text{ is the branch}$$

defined on $\mathbb{C} \setminus (-\infty, -1] \cup [1, +\infty)$

s.t. $f(z) > 0$ for $z = x > 1$



Find $f(0) = ?$, $f(i) = ?$



$z-1 = r_1 e^{i\theta_1}$, $\quad 0 \le \theta_1 < 2\pi$

$z+1 = r_2 e^{i\theta_2}$, $\quad -\pi \le \theta_2 \le \pi$

$$f(z) = \sqrt{r_1 r_2}\, e^{i\frac{\theta_1+\theta_2}{2}}$$

$$f(0) = \sqrt{1 \cdot 1} \cdot e^{i\frac{\pi+0}{2}} = e^{i\pi/2} = i$$

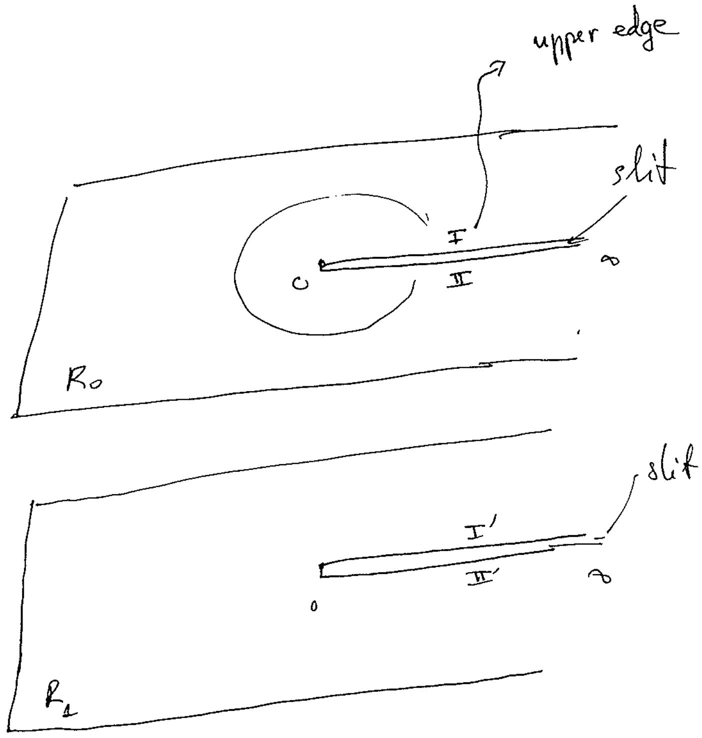
$$f(i) = \sqrt{\sqrt{2}\cdot\sqrt{2}}\, e^{i\frac{3\pi/4 + \pi/4}{2}} = \sqrt{2}\, e^{i\frac{\pi}{2}} = \sqrt{2}\, i$$

7 (§99). Riemann surfaces

a) Riemann surface of $\sqrt{z}$

$R_0 = \mathbb{C} \setminus [0, +\infty)$

$R_1 = \mathbb{C} \setminus [0, +\infty)$



The Rieman surface of $\sqrt{z}$

R is obtained by

$R_1 \cup R_1$ with $\mathrm{I} \equiv \mathrm{II}' = \mathrm{II}'$

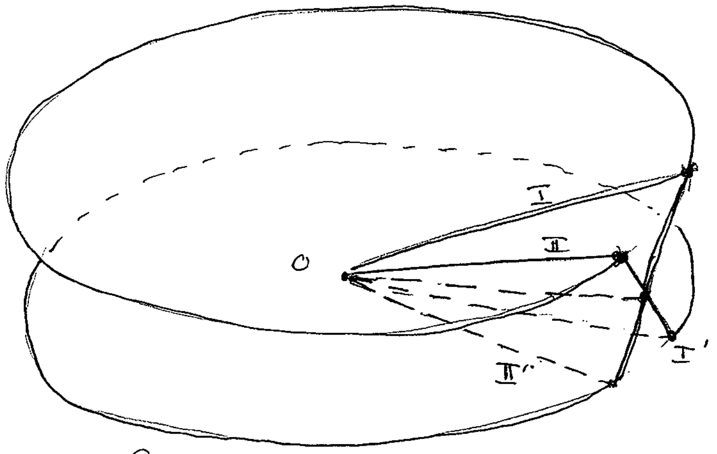
$\mathrm{II} \equiv \mathrm{I}' = \mathrm{I}'$

(I is joined to II'
II ______ I')

$\sqrt{z} = \sqrt{r}\, e^{i\theta/2}$

$\theta : 0 \longrightarrow 2\pi$

$\theta : 2\pi \longrightarrow 4\pi$



R

Remark

$$\mathbb{C} \xrightarrow{1-1} R \xrightarrow{1-1} \mathbb{C}$$

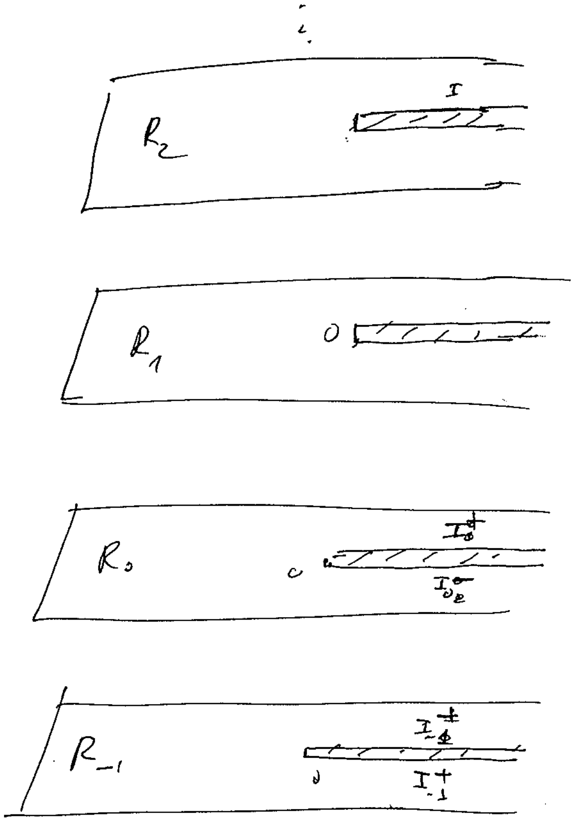
$$z \longmapsto \sqrt{z}$$

$$z \longmapsto z^2$$

b) Riemann surface of $\log z$

The Riemann surface (R) of $\log z$ is obtained by $\bigcup_{n=-\infty}^{+\infty} R_j$ with

$\mathrm{I}_n^- \equiv \mathrm{I}_{n+1}^+ \quad \forall n \in \mathbb{Z}$



$$I_n^+ \equiv I_{n+1}^-$$

8. Surfaces for related functions

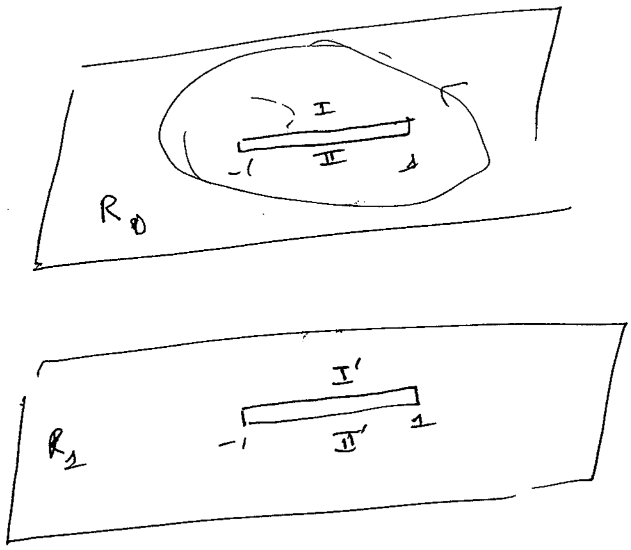
a) Riemann surface of $\sqrt{z^2-1}$ =

$$f(z) = \sqrt{r_1 r_2}\, \exp\left(\frac{i(\theta_1+\theta_2)}{2}\right), \quad z-1 = r_1 e^{i\theta_1}, \quad z+1 = r_2 e^{i\theta_2}$$

The Riemann surface R of $\sqrt{z^2-1}$ is $R_0 \cup R_1$ with

$\mathrm{I} \equiv \mathrm{II}'$ (I is joined to II')

$\mathrm{II} = \mathrm{I}'$



Chapter 9. Conformal mapping

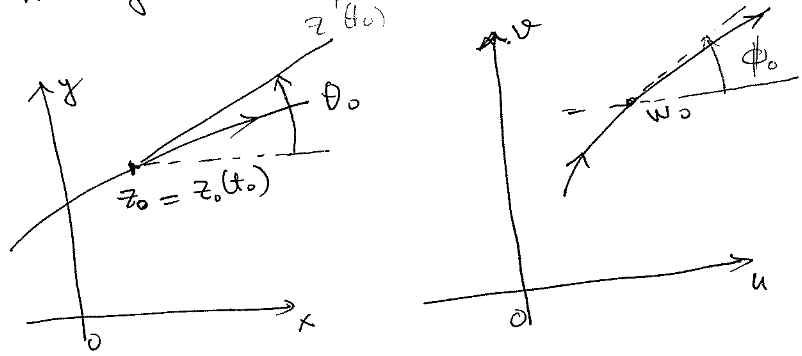
101. Preservation of angles

$\mathcal{C}: \quad z = z(t), \quad a \le t \le b, \quad f: \mathbb{C} \supset D \to \mathbb{C}$

$\Gamma = f(\mathcal{C}): \quad w = f[z(t)]$

$z_0 := z(t_0)$

$w_0 = f(z_0)$



$\theta_0 = \arg z_0'(t_0), \quad \phi_0 = \arg w_0'(t_0)$

$w_0'(t_0) = f'(z(t_0)) \cdot z_0'(t_0), \quad f'(z_0) \neq 0$

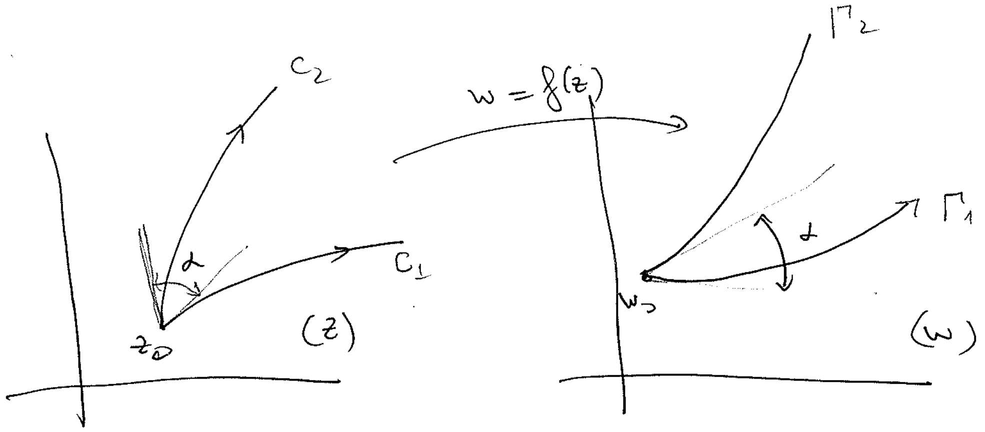
$\Rightarrow \arg w_0'(t_0) = \arg f'(z_0) + \arg z_0'(t_0)$

$\phi_0 = \psi_0 + \theta_0 \quad ; \quad \psi_0 = \arg f'(z_0)$ the angle of rotation.

$C_1: \quad z = z_1(t), \quad a \le t \le b$

$C_2: \quad z = z_2(t), \quad a' \le t \le b'$

$\Gamma_1 = f(C_1), \quad \Gamma_2 = f(C_2)$



$$\phi_1 = \psi_0 + \theta_1 \ , \ \phi_2 = \psi_0 + \theta_2$$

$$\hookrightarrow \quad \phi_2 - \phi_1 = \theta_2 - \theta_1 = \alpha$$

the angle $\phi_2 - \phi_1$ from Γ_1 to Γ_2 in magnitude & sense

the angle $\theta_2 - \theta_1$ from C_1 to C_2.

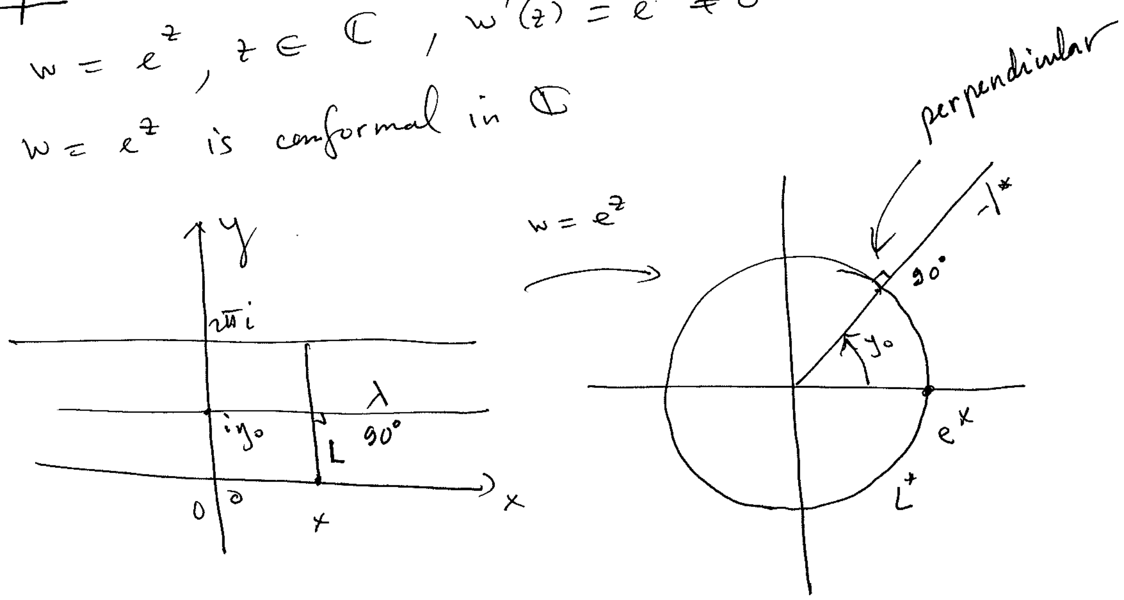
$\Rightarrow$ f is said to be "conformal" $\left(\exists f'(z_0) \neq 0 \right)$

__Def.__ A transformation $w = f(z)$, $z \in D$ is conformal in D if $f \in Hol(D)$ & $f'(z) \neq 0 \ \forall z \in D$.

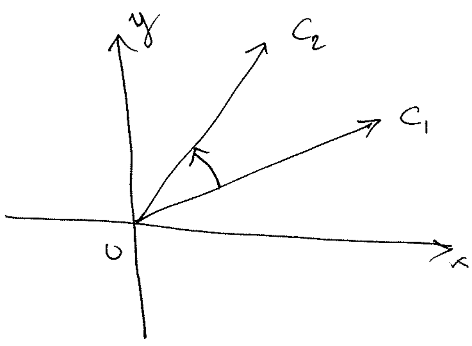
__Examples__

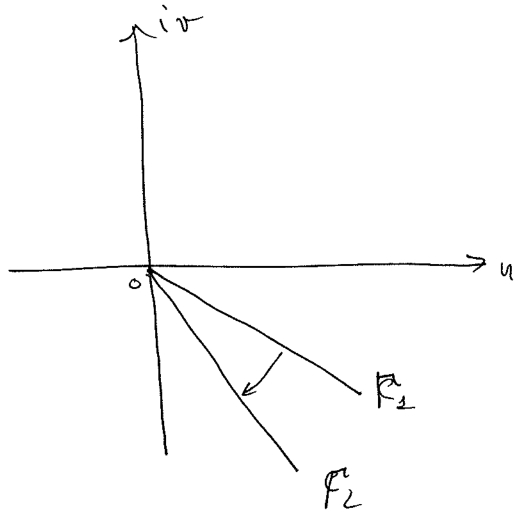
1) $w = e^z$, $z \in \mathbb{C}$, $w'(z) = e^z \neq 0 \ \forall z$

$w = e^z$ is conformal in $\mathbb{C}$



2) $w = \bar{z}$: is an isogonal mapping

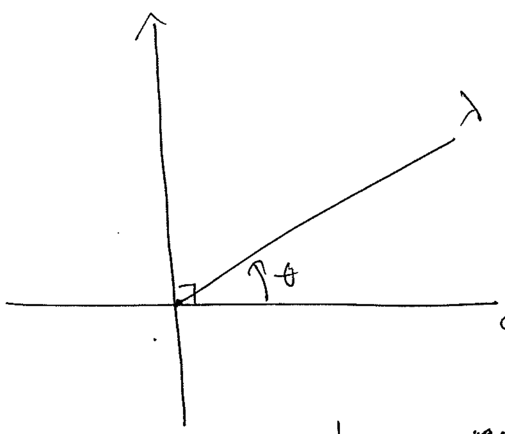


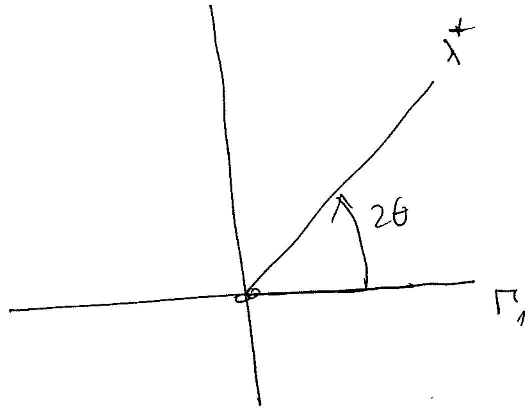


$$\phi_2 - \phi_1 = -(\theta_2 - \phi_1)$$

Def. A map that preserves the magnitude of the angle between two smooth curves but not necessary the sense is called an isogonal map.

3) $w = z^2$





$w = z^2$ does not ~~preserve~~ preserve at $z = 0$

but if preserves at any $z \neq 0$.

Scale factors

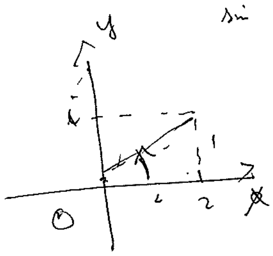
$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$\Rightarrow \left| f'(z_0) \right| = \lim_{z \to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|},$$

$$\Rightarrow \frac{|f(z) - f(z_0)|}{|z - z_0|} \approx |f'(z_0)|$$

$|f'(z_0)| > 1$, f is an expansion at z_0

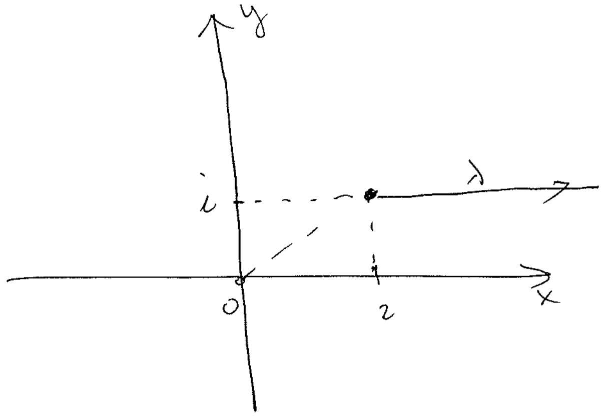
$|f'(z_0)| < 1$ f is a contraction at z_0

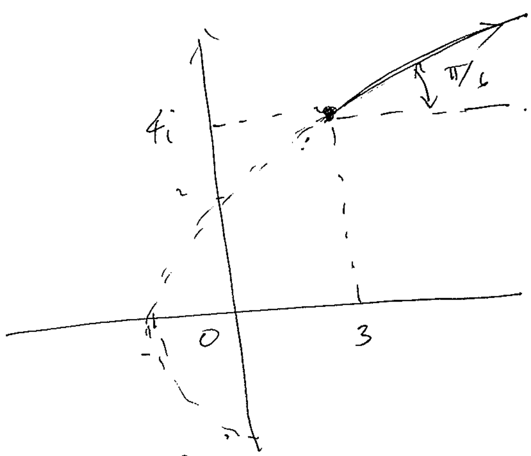


Example. $w = z^2$, $z_0 = 2 + i$

$$w'(z_0) = 2 \cdot z_0 = 4 + 2i$$

- $\varphi_0 = \arg w'(z_0) = \operatorname{ang}(4 + 2i) = \frac{\pi}{6} + 2n\pi$





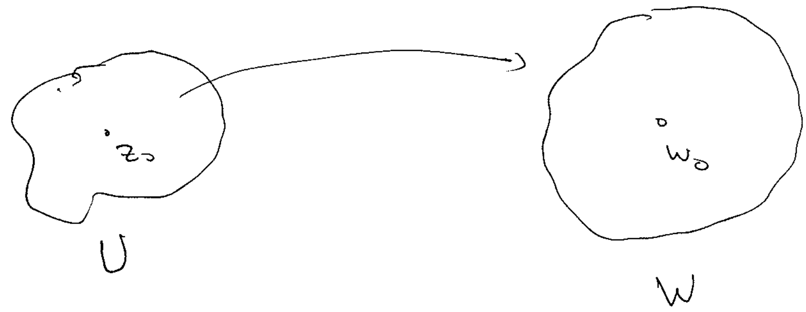
$z \in \lambda \Leftrightarrow z = x + i \Rightarrow w = z^2 = (x+i)^2$
$= x^2 - 1 + 2xi \Rightarrow u = \frac{v^2}{4} - 1$

- The scale factor of f at z_0 is $2\sqrt{5}$

$$|w'(z_0)| = \sqrt{4^2 + 2^2} = 4\sqrt{5}.$$

103. <u>Local inverses</u>

$$w = f(z) \quad , \quad f'(z_0) \neq 0$$



$\exists$ U is a nbd of z_0

W ——————— $w_0 = f(z_0)$ s.t

$$f : \quad U \xrightarrow{1-1} W$$

$$\Rightarrow \exists \, f^{-1} : W \longrightarrow U$$

$$w = f(z) \iff z = g(w) \Rightarrow f(g(w)) = w$$

$$\Rightarrow \boxed{g'(w) = \frac{1}{f'(z)}}$$

$$f = u + iv \quad , \quad z = x + iy$$

$$f : \mathbb{R}^2 \supset U \longrightarrow W \subset \mathbb{R}^2$$

$$(x, y) \longmapsto (u, v)$$

$$\begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases}$$

$$J_f(z) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} u_x & -v_x \\ v_x & u_x \end{vmatrix} = u_x^2 + v_x^2 = |f'(z)|^2.$$

Example $w = e^z$, $f'(z) = e^z$, $0 < \mathrm{Arg}\, w < 2\pi$

$$z = \log w = \ln|w| + i(\mathrm{Arg}\, w + 2n\pi),\ n \in \mathbb{Z}$$

$$g'(w) = \frac{1}{f'(z)} = \frac{1}{e^z} = \frac{1}{w}, \quad w \in \mathbb{C} \setminus \mathbb{R}^+$$

—

104. Harmonic Conjugates

- u, v are harmonic in $D \subset \mathbb{C}$
- v is a harmonic conjugate of u if

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \quad \text{(C-R equation)}$$

Fact. If D is simply-connected, then $\exists$ a harmonic conjugate of any harmonic function u in D.

Proof

$$v(x,y) = \int_{(x_0,y_0)}^{(x,y)} -u_t(s,t)\,ds + u_s(s,t)\,dt$$

$$dv = -u_y(x,y)\,dx + u_x(x,y)\,dy$$

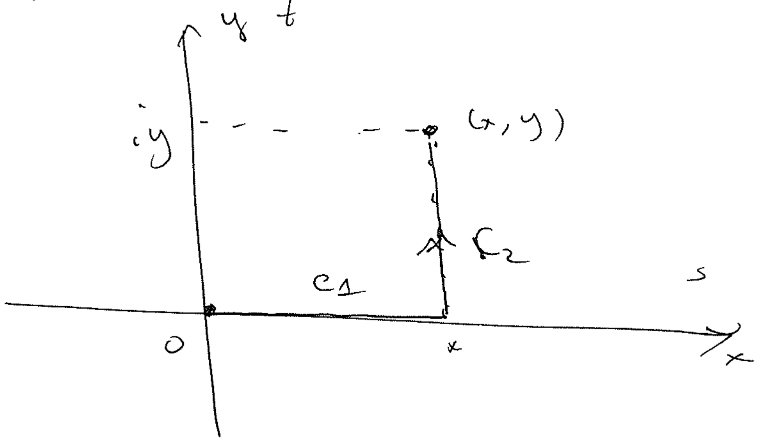
$$\begin{cases} v_x = -u_y \\ v_y = u_x \end{cases} \Rightarrow v \text{ is a harmonic conjugate of } u.$$

$\boxtimes$

Example

$u = x^2 - y^2 \quad , \quad u_x = 2x \, , \quad u_y = -2y$

$$v(x,y) = \int_{(0,0)}^{(x,y)} -2t\,ds + 2s\,dt$$



$$v = \int_{C_1} -2t\,ds + 2s\,dt + \int_{C_2} -2t\,ds + 2s\,dt$$

$$= \int_0^x \underbrace{-2t.0 + 0\,dt}_{=0} + \int_0^y +2x\,dt + 0$$

$$= +2xy$$

$v(x,y) = 2xy$ is a harmonic conjugate of $u = x^2 - y^2$

$f(z) = u + iv = z^2.$

105 Transformations of harmonic functions

Theorem. $w = f(z) = u(x,y) + i v(x,y) : D_z \xrightarrow{\text{onto}} \Omega_w$ is holomorphic

If $h : \Omega_w \longrightarrow \mathbb{R}$ is harmonic, then

$$H(x,y) := h[u(x,y), v(x,y)] \text{ is harmonic in } D_z.$$

Proof.

⊕ Ω_w is simply-connected.

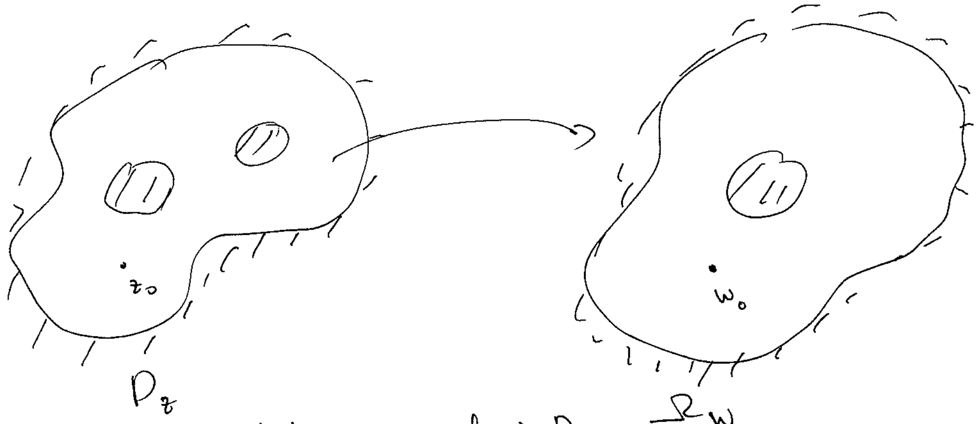
h is harmonic in Ω_w $\Rightarrow$ $\exists$ a harmonic conjugate of h, say, g.

$\Rightarrow$ $\phi(w) = h(u,v) + i\, g(u,v)$ is holomormic in Ω_w

$\Rightarrow$ $\phi \circ f$ is holomormic in D_z

$\Rightarrow$ $\mathrm{Re}(\phi \circ f) = h(u(x,y), v(x,y))$ is harmonic in D_z.

⊕ Ω_w is not simply-connected



Let $z_0 \in D_z$ be an arbitrary point in D_z

- $w_0 = f(z_0) \in \Omega_w \Rightarrow \exists\, \varepsilon > 0$ s.t. $D_\varepsilon(w_0) \subset \Omega_w$

- Since f is continuous, $\exists\, \delta > 0$ s.t.

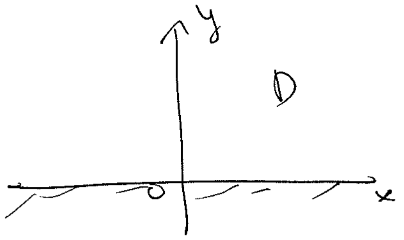
$$f(D_\delta(z_0)) \subset D_\varepsilon(w_0).$$

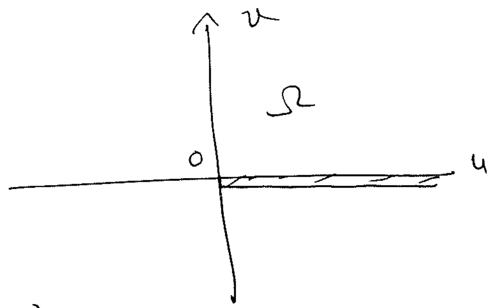
Since $D_\varepsilon(w_0)$ is simply-connected, $h(u(x,y) + i v(x,y))$ is harmonic in $D_\delta(z_0)$. Hence, $h(u(x,y) + i v(x,y))$ is harmonic in D_z.

<u>Examples</u>

1) - $h(u,v) = e^u \cos v$ is harmonic in $\mathbb{C}_w$

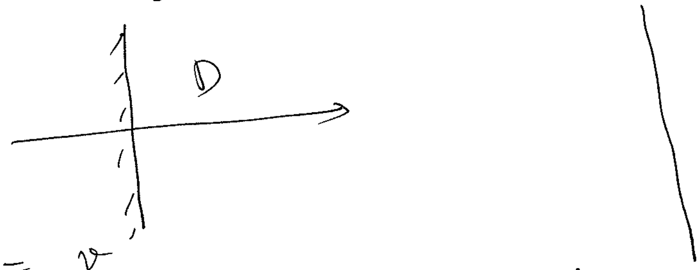
- $f(z) = z^2 : H^+ = \{\operatorname{Im} z > 0\} \longrightarrow \mathbb{C} \setminus R^+ = \Omega_w$





$$\Rightarrow H(x,y) = h(u(x,y), v(x,y))$$
$$= e^{x^2-y^2} \cos(2xy)) \text{ is harmonic in } D.$$

2) - $f(z) = \operatorname{Log} z = \ln|z| + i\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$



- $h = v$

$\Rightarrow H(x,y) = \operatorname{Im} f(x,y) = \theta = \arctan \frac{y}{x}$ is harmonic in $D = \{x > 0\}$.

109. Transformations of boundary conditions

Theorem. $w = f(z) = u(x,y) + iv(x,y)$ is conformal on a curve γ, $L = f(\gamma)$. If $h(u,v)$ satisfies either

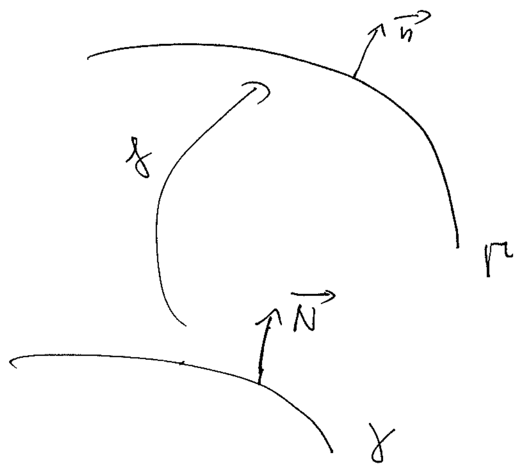
$$h = h_0 \quad \text{or} \quad \frac{dh}{dn} = 0 \quad \text{along } \Gamma, \text{ then}$$

$H(x,y) = h(u(x,y), v(x,y))$ satisfies

$$H = h_0 \quad \text{or} \quad \frac{dH}{dN} = 0 \quad \text{along } \gamma.$$

Note. $$\frac{dh}{d\vec{n}}(w_0) = \lim_{t \to 0} \frac{h(w_0 + \vec{n}\cdot t) - h(w_0)}{t} = (\overrightarrow{\text{grad } h}) \cdot \vec{n}$$

$$= \frac{\partial h}{\partial u} \cos\alpha + \frac{\partial h}{\partial v} \sin\alpha, \quad \vec{n} = (\cos\alpha, \sin\alpha)$$



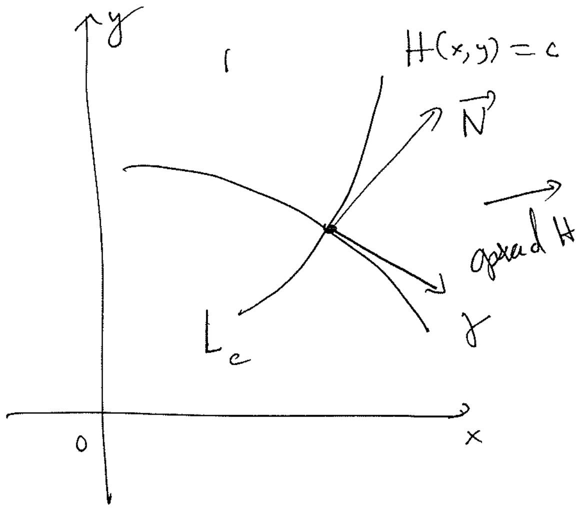
Proof.

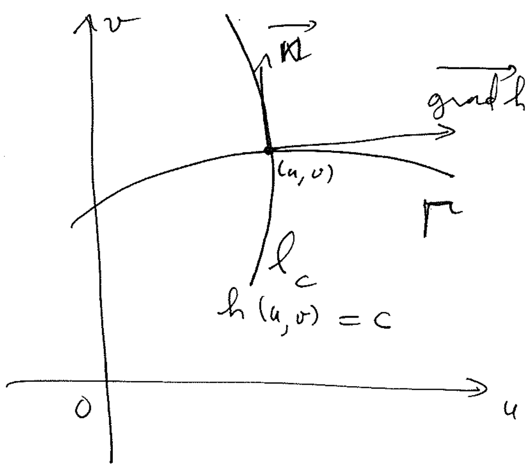
⊕ $h = h_0 \Rightarrow H(x,y) = h(u(x,y), v(x,y)) = h_0$ on γ

⊕ $\frac{\partial h}{\partial n} = 0$ along Γ, $\frac{\partial h}{\partial \vec{n}} = \overrightarrow{\text{grad } h} \cdot \vec{n} = 0$

$\Rightarrow \overrightarrow{\text{grad } h} \perp \vec{n}$ ($\overrightarrow{\text{grad } h}$ is orthogonal to $\vec{n}$ at (u,v))

We will show that $\overrightarrow{\text{grad } H} \perp \vec{N}$.





- $z \in \gamma \longrightarrow w = f(z) \in \Gamma$

- $l_c = \{(u,v) \in \mathbb{C} \mid h(u,v) = c\}$ the level set of h

- $L_c = \{(x,y) \in \mathbb{C} \mid H(x,y) = c\}$

Since $H(x,y) = h(u(x,y), v(x,y)) = c$,

l_c is the image of L_c, L_c is transformed into l_c

$$L_c \xrightarrow{\text{by } f} l_c$$

$$\gamma \longrightarrow \Gamma$$

Since, $\overrightarrow{\text{grad } H} \perp L_c \Rightarrow \overrightarrow{\text{grad } H}$ is tangent to γ

$\left(\overrightarrow{\text{grad } h} \perp l_c \Rightarrow \overrightarrow{\text{grad } h} \text{ is tangent to } \Gamma \right)$

$$\Rightarrow \quad \overrightarrow{\text{grad } H} \perp \vec{N}$$

$$\Rightarrow \quad \frac{dH}{dN} = \overrightarrow{\text{grad } H} \cdot \vec{N} = 0$$

Note: if $\overrightarrow{\text{grad } h} = 0 \Rightarrow |\overrightarrow{\text{grad } H}| = |\overrightarrow{\text{grad } h}| \cdot |f'(z)| = 0$

$\Leftarrow\!\!\!\!\!/ \quad \overrightarrow{\text{grad } H} = 0 \ \boxtimes$

Homework I (due to Tuesday 26 Nov., 2013)

Math 210: Applied Complex Varibles, Fall 2013

1. Let C_ρ be the circle $\{z \in \mathbb{C} \mid |z| = \rho\}$ $(\rho > 0)$, described in the counterclockwise direction. Find the values of the following integerals:

   (a) (20 points)
$$\int_{C_2} \frac{z^2 + 2013z + 2014}{z^{2013} + 1} dz$$

   (b) (20 points)
$$\int_{C_2} \frac{e^z}{z(z+1)(z+3)} dz$$

   (c) (10 points)
$$\int_{C_3} (1 + z^2)(e^{\frac{1}{z}} + e^{\frac{1}{z-1}} + e^{\frac{1}{z-2}}) dz.$$

2. Calculate the following integrals:

   (a) (20 points)
$$\int_0^{+\infty} \frac{x^2 + 3}{x^4 + x^2 + 1} dx$$

   (b) (20 points)
$$\int_0^{+\infty} \left(\frac{\sin(x)}{x}\right)^2 dx$$

   (c) (20 points)
$$\int_0^{+\infty} \frac{x^a}{x^2 + 1} dx$$

   , where $-1 < a < 1$

   (d) (20 points)
$$\int_0^{+\infty} \frac{\ln(x)}{x^2 + x + 1} dx$$

   (e) (10 points)
$$\int_0^{+\infty} \frac{\sin(x)}{x(x^2 + 1)} dx.$$

End

Solutions for Homework I

11/25/2013

1 a)

$$\oint_{|z|=2} \frac{z^2+2013z+2014}{z^{2013}+1}\,dz = -2\pi i \operatorname*{Res}_{z=\infty} f(z)$$

$$= -2\pi i \lim_{z\to\infty}\left(z\cdot\frac{z^2+2013z+2014}{z^{2013}+1}\right) = 0$$

b)

$$\oint_{|z|=2} \frac{e^z}{z(z+1)(z+3)}\,dz = 2\pi i\left[\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=-1} f(z)\right]$$

$$= 2\pi i\left[\left.\frac{e^z}{(z+1)(z+3)}\right|_{z=0} + \left.\frac{e^z}{z(z+3)}\right|_{z=-1}\right]$$

$$= 2\pi i\left[\frac{1}{3} + \frac{e^{-1}}{(-1)\cdot 2}\right] = 2\pi i\left[\frac{1}{3} + \frac{-1}{2e}\right]$$

c)

$$I = \int_{|z|=3} (1+z^2)\left(e^{\frac{1}{z}} + e^{\frac{1}{z-1}} + e^{\frac{1}{z-2}}\right) dz$$

$$= \int_{|z|=3} (1+z^2)\, e^{\frac{1}{z}}\,dz + \int_{|z|=3} (1+z^2)\, e^{\frac{1}{z-1}}\,dz + \int_{|z|=3} (1+z^2)\, e^{\frac{1}{z-2}}\,dz$$

$$= 2\pi i\left[\operatorname*{Res}_{z=0}\left[(1+z^2)\,e^{\frac{1}{z}}\right] + \operatorname*{Res}_{z=1}\left[(1+z^2)\,e^{\frac{1}{z-1}}\right] + \operatorname*{Res}_{z=2}\left[(1+z^2)\,e^{\frac{1}{z-2}}\right]\right].$$

$+\quad (1+z^2)\,e^{\frac{1}{z}} = (1+z^2)\left(1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \cdots\right)$

$$\Rightarrow \operatorname*{Res}_{z=0} (1+z^2)\,e^{\frac{1}{z}} = 1 + \frac{1}{3!} = \frac{7}{6}$$

$+\quad (1+z^2)\,e^{\frac{1}{z-1}} = \left[1+((z-1)+1)^2\right]\left[1 + \frac{1}{z-1} + \frac{1}{2!\,(z-1)^2} + \frac{1}{3!\,(z-1)^3} + \cdots\right.$

$$= \left(2 + 2(z-1) + (z-1)^2\right)\left(1 + \frac{1}{z-1} + \frac{1}{2(z-1)^2} + \frac{1}{6(z-1)^3} + \cdots\right)$$

$$\Rightarrow \operatorname*{Res}_{z=1}\left[(1+z^2)e^{\frac{1}{z-1}}\right] = 2 + 2\cdot\frac{1}{2} + \frac{1}{6} = 3 + \frac{1}{6} = \frac{19}{6}$$

$$+ \quad (1+z^2)\, e^{\frac{1}{z-2}} = \left[1 + (z-2+2)^2\right]\left[1 + \frac{1}{z-2} + \frac{1}{2(z-2)^2} + \frac{1}{6(z-2)^3} + \cdots\right]$$

$$= \left[5 + 4(z-2) + (z-2)^2\right]\left[1 + \frac{1}{z-2} + \frac{1}{2(z-2)^2} + \frac{1}{6(z-2)^3} + \cdots\right]$$

$$\Rightarrow \operatorname*{Res}_{z=1}\left[(1+z^2)e^{\frac{1}{z-2}}\right] = 5 + 4\cdot\frac{1}{2} + \frac{1}{6} = 7 + \frac{1}{6}$$

$$= \frac{43}{6}$$

Thus, $I = 2\pi i\left[\frac{7}{6} + \frac{19}{6} + \frac{43}{6}\right)$

$$= 2\pi i\left[11 + \frac{1}{2}\right] = 23\pi i$$

2. a) $$I = \int_0^{+\infty} \frac{x^2+3}{x^4+x^2+1}\,dx = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{x^2+3}{x^4+x^2+1}\,dx$$

$$= \frac{1}{2}\cdot 2\pi i\left[\operatorname*{Res}_{z=z_1} \frac{z^2+3}{z^4+z^2+1} + \operatorname*{Res}_{z=z_2} \frac{z^2+3}{z^4+z^2+1}\right]$$

$$(z^2-1)(z^4+z^2+1) = z^6 - 1 = 0 \Leftrightarrow z^6 = 1 = e^0$$

$$\Leftrightarrow \quad z_k = e^{\frac{2k\pi}{6}}, \quad k = 0, 1, 2, 3, 4, 5$$
$$= e^{k\pi/3}$$

$\Rightarrow \quad z_1 = e^{\pi/3}, \; z_2 = e^{2\pi/3}$ are ~~sol~~ simple zeros

of $z^4 + z^2 + 1$.

$$\Rightarrow \operatorname*{Res}_{z=z_k} \frac{z^2+3}{z^4+z^2+1} = \operatorname*{Res}_{z=z_k} \frac{(z^2-1)(z^2+3)}{z^6-1} = \frac{(z_k^2-1)(z_k^2+3)}{6z_k^5}$$

$$= z_k\frac{(z_k^2-1)(z_k^2+3)}{6}, \quad k=1,2$$

$$\Rightarrow I = \frac{\pi i}{6}\left[z_k(z_1^2-1)(z_1^2+3) + (z_2^2-1)(z_2^2+3)\right]$$

$$= \frac{\pi i}{6}\left[z_1(z_1^4+2z_1^2-3) + z_2(z_2^4+2z_2^2-3)\right]$$

$$= \frac{\pi i}{6}\left[z_1\Big(\underbrace{z_1^4+z_1^2+1}_{=0} + (z_1^2-4)\Big) + z_2\Big(\underbrace{z_2^4+z_2^2+\cancel{1}}_{=0} + z_2^2-4\Big)\right]$$

$$= \frac{\pi i}{6}\left[z_1^3 + z_2^3 - 4(z_1+z_2)\right]$$

$$= \frac{\pi i}{6}\left[-1+1-4\left(e^{\frac{\pi i}{3}} + e^{\frac{2\pi i}{3}}\right)\right]$$

$$= \frac{\pi i}{6}\cdot(-4)\,2i\sin\frac{\pi}{3} = \cdot\frac{2\pi}{3}\sqrt{3} = \frac{2\pi}{\sqrt{3}}.$$

b) $$I = \int_0^{+\infty}\left(\frac{\sin x}{x}\right)^2 dx = \frac{1}{2}\int_{-\infty}^{+\infty}\frac{1-\cos(2x)}{2x^2}\,dx$$

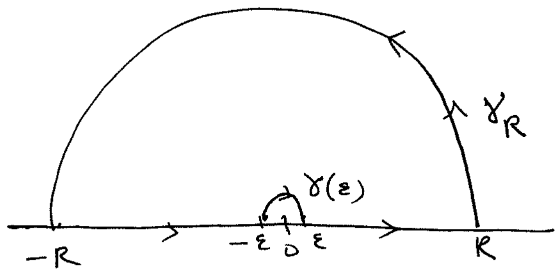
$$= \frac{1}{4}\int_{-\infty}^{+\infty}\frac{1-\cos 2x}{x^2}\,dx = \frac{1}{4}\operatorname{Re} P.V.\int_{-\infty}^{+\infty}\frac{1-e^{2ix}}{x^2}\,dx$$

Choose $f(z) = \dfrac{1-e^{2iz}}{z^2}$.

Then $$f(z) = \frac{1-(1+2iz+\cdots)}{z^2} = -\frac{2i}{z} + \varphi(z)$$

$$\Rightarrow \operatorname*{Res}_{z=0} f(z) = -2i$$

Choose $\Gamma(R,\varepsilon) = [-R,-\varepsilon] \cup \gamma_\varepsilon \uplus [\varepsilon, R] \cup \gamma_R$



$$\lim_{\varepsilon \to 0^+} \int_{\gamma_\varepsilon} f(z)\,dz = -\pi i \operatorname{Res}_{z=0} f(z) = -\pi i(-2i) = -2\pi$$

$$0 \le \left| \int_{\gamma_R} \frac{1-e^{2iz}}{z^2}\,dz \right| \le \frac{1+1}{R^2} \cdot \pi R$$

$$0 \Rightarrow \lim_{R\to+\infty} \int_{\gamma_R} f(z)\,dz = 0$$

By Cauchy - Goursat's theorem, we have

$$0 = \int_{\Gamma(R,\varepsilon)} f(z)\,dz = \left[\int_{-R}^{-\varepsilon} f(x)\,dx + \int_{\varepsilon}^{R} f(x)\,dx\right] + \int_{\gamma_\varepsilon} f(z)\,dz + \int_{\gamma_R} f(z)\,dz$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$\text{P.V.} \int_{-\infty}^{+\infty} f(x)\,dx \qquad -2\pi \qquad 0$$

Let $R \to +\infty$ and $\varepsilon \to 0^+$ we obtain that

$$\text{P.V.} \int_{-\infty}^{+\infty} f(x)\,dx = 2\pi \Rightarrow \int_0^{+\infty} \left(\frac{\sin x}{x}\right)^2 dx = \frac{2\pi}{4} = \frac{\pi}{2}.$$

c) $\displaystyle I = \int_0^{+\infty} \frac{x^a}{x^2+1}\,dx \qquad (-1 < a < 1)$

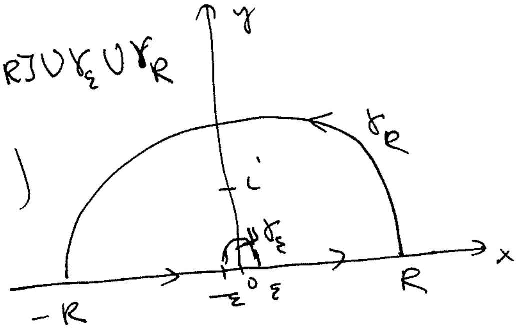
Choose $\displaystyle f(z) = \frac{z^a}{z^2+1} = \frac{e^{a\,\mathrm{Log}\,z}}{z^2+1}$

$$= \frac{e^{a(\ln|z| + i\,\mathrm{Arg}\,z)}}{z^2+1}, \quad 0 \le \mathrm{Arg}\,z \le 2\pi$$

$$\operatorname*{Res}_{z=i} f(z) = \frac{e^{a\,\mathrm{Log}\,i}}{2i} = \frac{e^{\frac{a\pi}{2}i}}{2i} = \frac{1}{2i}\, e^{\frac{a\pi i}{2}}$$

Choose $\Gamma(R,\varepsilon) = [-R,-\varepsilon] \cup [\varepsilon, R] \cup \gamma_\varepsilon \cup \gamma_R$

$(R > 1 > \varepsilon > 0)$



$$0 \le \left| \int_{\gamma_R} f(z)\,dz \right| \le \frac{R^a}{R^2-1}\,\pi R$$

0

$$\Rightarrow \int_{\gamma_R} f(z)\,dz \to 0 \text{ as } R \to +\infty$$

$$0 \le \left| \int_{\gamma_\varepsilon} f(z)\,dz \right| \le \frac{\varepsilon^a \cdot \pi\varepsilon}{1-\varepsilon^2} \Rightarrow \int_{\gamma_\varepsilon} f(z)\,dz \to 0 \text{ as } R \to +\infty.$$

0

$$f(z)\Big|_{z=-x\,\in\,[-R,-\varepsilon]} = \frac{e^{a(\ln x + i\pi)}}{x^2+1} = \frac{x^a e^{ia\pi}}{x^2+1}$$

$$\int_{[-R,-\varepsilon]} f(z)\,dz = \int_R^{\varepsilon} \frac{x^a e^{ia\pi}}{x^2+1}(-dx) = \int_\varepsilon^R \frac{x^a e^{ia\pi}}{x^2+1}\,dx$$

By Cauchy's residue theorem, we get

$$2\pi i \operatorname*{Res}_{z=i} f(z) = \int_{\Gamma(R,\varepsilon)} f(z)\,dz = \int_{[-R,-\varepsilon]} f(z)\,dz + \underbrace{\int_{\gamma(\varepsilon)} f(z)\,dz}_{\xrightarrow{\varepsilon\to 0^+} 0} + \int_{[\varepsilon,R]} f(x)\,dx + \underbrace{\int_{\gamma_R} f(z)\,dz}_{\xrightarrow{R\to+\infty} 0}$$

$$\Rightarrow 2\pi i \frac{e^{\frac{a\pi i}{2}}}{2i} = e^{a\pi i}\int_{\varepsilon}^{R} \frac{x^a}{x^2+1}\,dx + \int_{\varepsilon}^{R} \frac{x^a}{x^2+1}\,dx + \int_{\gamma_\varepsilon} f(z)\,dz + \int_{\gamma_R} f(z)\,dz$$

Let $R \to +\infty$ & $\varepsilon \to 0^+$ we obtain that

$$\pi e^{\frac{a\pi i}{2}} = \left(1 + e^{a\pi i}\right)\int_0^{+\infty} \frac{x^a}{x^2+1}\,dx$$

$$\Rightarrow \int_0^{+\infty} \frac{x^a}{x^2+1}\,dx = \pi \cdot \frac{1}{e^{-\frac{a\pi i}{2}} + e^{\frac{a\pi i}{2}}} = \frac{\pi}{2\cos\frac{a\pi}{2}}$$

$$= \frac{\pi}{2\cos\frac{a\pi}{2}}$$

d) $I = \displaystyle\int_0^{+\infty} \frac{\ln x}{x^2+x+1}\,dx$

Choose $f(z) = \dfrac{(\mathrm{Log}\, z)^2}{z^2+z+1} = \dfrac{(\ln|z| + i\,\mathrm{Arg}\, z)^2}{z^2+z+1}$, $0 \le \mathrm{Arg}\, z \le 2\pi$

$z^2+z+1=0 \Rightarrow z_1 = \dfrac{-1+i\sqrt{3}}{2} = e^{2\pi i/3}$

$z_2 = \dfrac{-1-i\sqrt{3}}{2} = e^{4\pi i/3}$

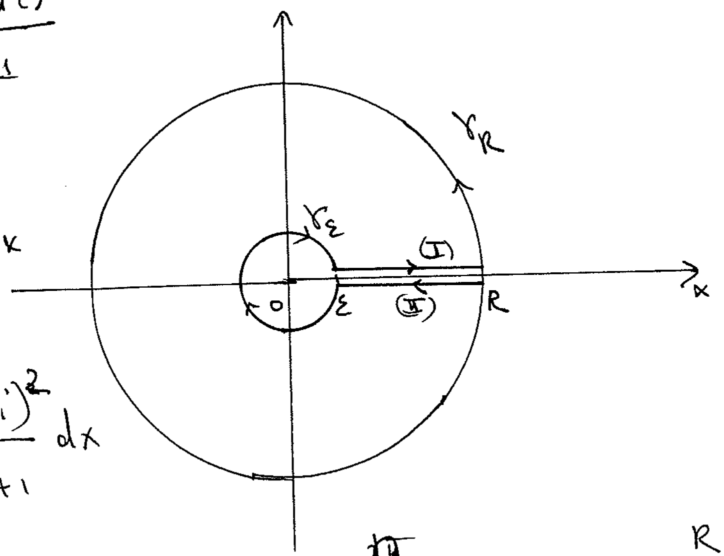
$\displaystyle\operatorname*{Res}_{z=z_1} f(z) = \frac{(\mathrm{Log}\, z_1)^2}{2z_1+1} = \frac{(\mathrm{Log}\, e^{2\pi i/3})^2}{i\sqrt{3}} = -\frac{(2\pi i)^2}{i9\sqrt{3}} = -\frac{4\pi^2}{i9\sqrt{3}}$

$\displaystyle\operatorname*{Res}_{z=z_2} f(z) = \frac{(\mathrm{Log}\, z_2)^2}{2z_2+1} = \frac{\left(\frac{4\pi i}{3}\right)^2}{-i\sqrt{3}} = +\frac{16\pi^2}{i9\sqrt{3}}$

Choose $\Gamma(R,\varepsilon) = (\mathrm{I}) \cup (\mathrm{II}) \cup \gamma_\varepsilon \cup \gamma_R$, $\gamma_\varepsilon = \{|z| = \varepsilon\}$

$\gamma_R = \{|z| = R\}$

$\displaystyle f(z)\Big|_{z=x\in(\mathrm{II})} = \frac{(\ln x + 2\pi i)^2}{x^2+x+1}$



$\displaystyle\int_{(\mathrm{II})} f(z)\,dz = \int_R^\varepsilon \frac{(\ln x + 2\pi i)^2}{x^2+x+1}\,dx$

$\displaystyle = -\int_\varepsilon^R \frac{(\ln x + 2\pi i)^2}{x^2+x+1}\,dx$

$\displaystyle = -\int_\varepsilon^R \frac{(\ln x)^2}{x^2+x+1} - 4\pi i \int_\varepsilon^R \frac{\ln x}{x^2+x+1} + 4\pi^2 \int_\varepsilon^R \frac{dx}{x^2+x+1}$

$\displaystyle\int_{(\mathrm{I})} f(z)\,dz = \int_\varepsilon^R \frac{(\ln x)^2}{x^2+x+1}\,dx$

$$0 \le \left| \int_{\gamma_\varepsilon} f(z)\,dz \right| \le \frac{(\ln \varepsilon)^2 + (2\pi)^2}{1 - \varepsilon - \varepsilon^2} \cdot \pi\varepsilon$$

$$0 \qquad\qquad 0$$

$$\Rightarrow \int_{\gamma_\varepsilon} f(z)\,dz \to 0 \text{ as } \varepsilon \to 0^+$$

$$0 \le \left| \int_{\gamma_R} f(z)\,dz \right| \le \frac{(\ln R)^2 + (2\pi)^2}{R^2 - R - 1} \cdot \pi R$$

$$0 \qquad\qquad 0$$

$$\Rightarrow \int_{\gamma_R} f(z)\,dz \to 0 \text{ as } R \to +\infty$$

By Cauchy' residue theorem, we get

$$2\pi i \left(\operatorname*{Res}_{z=z_1}(f) + \operatorname*{Res}_{z=z_2}(f) \right) = \int_{\Gamma(R,\varepsilon)} f(z)\,dz = \int_{\gamma_R} f\,dz + \int_{\gamma_\varepsilon} f(z)dz + \int_{(I)} f dz + \int_{(II)} f(z)\,dz$$

$$\Leftrightarrow 2\pi i \left(-\frac{4\pi^2}{i9\sqrt{3}} + \frac{16\pi^2}{i9\sqrt{3}} \right) = \int_{\gamma_R} f(z)\,dz + \int_{\gamma_\varepsilon} f(z)\,dz + \int_{(I)} f(z)\,dz + \int_{(II)} f(z)\,dz$$

Let $R \to +\infty$ and $\varepsilon \to 0^+$, we obtain that

$$\Rightarrow \frac{8\pi^3}{3\sqrt{3}} = -4\pi i \int_0^{+\infty} \frac{\ln x}{x^2+x+1}\,dx + 4\pi^2 \int_0^{+\infty} \frac{dx}{x^2+x+1}$$

$$\Rightarrow \begin{cases} \displaystyle\int_0^{+\infty} \frac{\ln x}{x^2+x+1}\,dx = 0 \\[2ex] \displaystyle\int_0^{+\infty} \frac{dx}{x^2+x+1} = \frac{2\pi}{3\sqrt{3}} \end{cases}$$

e) $$I = \int_0^{+\infty} \frac{\sin x}{x(x^2+1)}\,dx = \frac{1}{2}\,\mathrm{Im}\ \text{P.V.}\int_{-\infty}^{+\infty} \frac{e^{ix}}{x(x^2+1)}\,dx$$

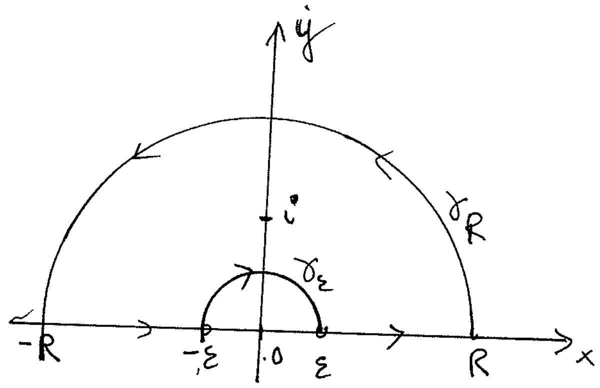
Choose $$f(z) = \frac{e^{iz}}{z(z^2+1)}$$

$$\operatorname*{Res}_{z=0} f(z) = \left.\frac{e^{iz}}{z^2+1}\right|_{z=0} = 1$$

$$\operatorname*{Res}_{z=i} f(z) = \left.\frac{e^{iz}}{z(z+i)}\right|_{z=i} = \frac{e^{-1}}{i\cdot 2i} = -\frac{1}{2e}$$

Choose $\Gamma(R,\varepsilon) = [-R,-\varepsilon] \cup \gamma(\varepsilon) \cup [\varepsilon, R] \cup \gamma_R$

$\gamma_\varepsilon = \gamma(\varepsilon) = \{ |z| = \varepsilon,\ \mathrm{Im}\, z \geq 0 \}$, $\gamma_R = \{ |z| = R,\ \mathrm{Im}\, z \geq 0 \}$



$$\int_{\gamma_\varepsilon} f(z)\,dz \Rightarrow -\pi i \operatorname*{Res}_{z=0} f = -\pi i \quad \text{as } \varepsilon \to 0^+$$

(By Theorem in Sec. 82 p. 277)

$$\int_{\gamma_R} f(z)\,dz = \int_{\gamma_R} \frac{e^{iz}}{z(z^2+1)}\,dz \to 0 \quad \text{as } R \to +\infty$$

(by Jordan's lemma)

By Cauchy's ~~res~~ residue theorem, we have

$$2\pi i \operatorname*{Res}_{z=i}(f) = \int_{\Gamma(R,\varepsilon)} f(z)\,dz = \underbrace{\int_{\gamma_\varepsilon} f(z)\,dz}_{\downarrow\ -\pi i} + \underbrace{\int_{\gamma_R} f(z)\,dz}_{\downarrow\ R\to+\infty\ \ 0} + \underbrace{\int_{-R}^{-\varepsilon} f(x)\,dx + \int_{\varepsilon}^{R} f(x)\,dx}_{\downarrow\ R\to+\infty,\ \varepsilon\to 0^+\ \ \text{P.V.}\int_{-\infty}^{+\infty}\frac{e^{ix}}{x(x^2+1)}dx}$$

Let $R \to +\infty$, $\varepsilon \to 0^+$ we obtain that

$$2\pi i\left(-\frac{1}{2e}\right) = -\pi i + \text{P.V.} \int_{-\infty}^{+\infty} \frac{e^{ix}}{x(x^2+1)}\,dx$$

$$\Leftrightarrow \text{P.V.} \int_{-\infty}^{+\infty} \frac{e^{ix}}{x(x^2+1)}\,dx = i\pi\left(1 - \frac{1}{e}\right)$$

So,
$$\int_0^{+\infty} \frac{\sin x}{x(x^2+1)}\,dx = \frac{1}{2}\,\mathrm{Im}\,\text{P.V.}\int_{-\infty}^{+\infty} \frac{e^{ix}}{x(x^2+1)}\,dx$$

$$= \frac{1}{2}\cdot\pi\left(1 - \frac{1}{e}\right).$$

Homework II (due to Tuesday 3 Dec., 2013)

Math 210: Applied Complex Varibles, Fall 2013 .

1. Find the poles and residues of the following functions:

   (a) (10 points)
   $$f_1(z) = \frac{1}{(z^2-1)^2}$$

   (b) (10 points)
   $$f_2(z) = \frac{1}{\sin(z)}$$

   (c) (10 points)
   $$f_3(z) = \frac{1}{\sin^2(z)}$$

   (d) (5 points)
   $$f_4(z) = \frac{1}{e^z-1} - \frac{1}{z}.$$

2. Find the values of the following integerals:

   (a) (10 points)
   $$\int_0^{2\pi} \frac{dx}{\cos(x)+\sin(x)+2}$$

   (b) (10 points)
   $$\int_0^{\pi} \frac{\cos(2x)}{1+2a\cos(x)+a^2}dx \ (-1 < a < 1)$$

   (c) (5 points)
   $$\int_0^{\pi} \cos^{2n}(x)dx \ (n = 1, 2, \ldots)$$

   (d) (5 points)
   $$\int_0^{2\pi} \sin^{2n}(x)dx \ (n = 1, 2, \ldots).$$

3. Determine the number of zeros, counting multiplicities, of the following polynomials:

   (a) (10 points) $z^{2014} - 6z^{2013} + 3z^{2012} - 1$ in $\{z \in \mathbb{C} \mid |z| < 1\}$

   (b) (10 points) $z^3 + 4z^2 - 2z + 2$ in $\{z \in \mathbb{C} \mid |z| < 3\}$.

4. Determine the number of roots, counting multiplicities, of the following equations:

   (a) (10 points) $2z^5 - 5z^2 + z + 1 = 0$ in the annulus $\{z \in \mathbb{C} \mid 1 < |z| < 2\}$

   (b) (5 points) $0.9e^{-z} + 1 = 2z$ in the domain $\{z \in \mathbb{C} \mid |z| < 1, \ \mathrm{Re}z > 0\}$.

——————End——————

11/29/2013

Solutions / answers for Homework II

1 a) $$f_1(z) = \frac{1}{(z^2-1)^2} = \frac{1}{(z-1)^2 \cdot (z+1)^2}$$

- $z_1 = 1$, $z_2 = -1$ are poles of order 2.

- $$\operatorname{Res}_{z=1} f_1 = \left[\frac{1}{(z+1)^2}\right]'_{z=1} = -\frac{2}{(z+1)^3}\Bigg|_{z=1} = -\frac{1}{4}$$

- $$\operatorname{Res}_{z=-1} f_1 = \left[\frac{1}{(z-1)^2}\right]'_{z=-1} = -\frac{2}{(z-1)^3}\Bigg|_{z=-1} = \frac{1}{4}$$

b) $$f_2(z) = \frac{1}{\sin z}$$

- $\sin z = 0 \iff z = n\pi$, $n \in \mathbb{Z}$

- Since $(\sin z)'_{z=n\pi} = \cos(n\pi) = (-1)^n \neq 0$,

$z = n\pi$ is a simple zero of $\sin z$ and thus $z = n\pi$ is a simple pole of f_2.

- $$\operatorname{Res}_{z=n\pi} f_2 = \frac{1}{(\sin z)'_{z=n\pi}} = \frac{1}{\cos n\pi} = (-1)^n.$$

c) $$f_3(z) = \frac{1}{\sin^2(z)}$$

- $z = n\pi$ is a pole of order 2 of f_2.

- $$\operatorname{Res}_{z=n\pi} f_3 = \frac{1}{1!}\left[\frac{(z-n\pi)^2}{\sin^2(z)}\right]'_{z=n\pi} =$$

- $$= \frac{2(z-n\pi)\sin^2(z) - (z-n\pi)^2 \cdot 2\sin z\cos z}{\sin^4 z}\Bigg|_{z=n\pi}$$

$$= \frac{2(z-n\pi)\left[\sin z - (z-n\pi)\cos(z)\right]}{\sin^3(z)}\Bigg|_{z=n\pi}$$

$$= \frac{2(z-n\pi)\left[(-1)^n\sin(z-n\pi) - (-1)^n(z-n\pi)\cos(z-n\pi)\right]}{(-1)^{3n}\sin^3(z-n\pi)}\Bigg|_{z=n\pi}$$

$$\overset{t:=z-n\pi}{=\!=} \; 2\cdot\frac{t\cdot(\sin t - t\cos t)}{\sin^3 t}\Bigg|_{t=0}$$

$$= \frac{2t\left[t-\frac{t^3}{6}+o(t^3) - t\left(1-\frac{t^2}{2}+o(t^2)\right)\right]}{\sin^3 t}\Bigg|_{t=0}$$

$$= \lim_{t\to 0}\frac{\frac{2}{3}t^4 + o(t^4)}{\sin^3(t)} = 0$$

(Note: $\sin(z) = \sin(z-n\pi+n\pi) = (-1)^n\sin(z-n\pi)$

$\cos(z) = \cos(z-n\pi+n\pi) = (-1)^n\cos(z-n\pi)$).

d) $$f_4(z) = \frac{1}{e^z-1} - \frac{1}{z}$$

- $e^z - 1 = 0 \iff z = 2n\pi i,\ n\in\mathbb{Z}$

- $(e^z-1)'_{z=2n\pi i} = e^z\Big|_{z=2n\pi i} = e^{2n\pi i} = 1 \neq 0$

$\Rightarrow$ $z = 2n\pi i$, ($n = \pm1, \pm2, \dots$) is a simple pole of f_4.

$$\operatorname*{Res}_{z=2n\pi i}(f_4) = \operatorname*{Res}_{z=2n\pi i}\frac{1}{e^z-1} - \underbrace{\operatorname*{Res}_{z=2n\pi i}\frac{1}{z}}_{=0}, \quad n\neq 0$$

$$= \frac{1}{(e^z-1)'\big|_{z=2n\pi i}} - 0$$

$$= \frac{1}{e^{2n\pi i}} - 0 = 1.$$

$\ast$ $z=0$ is a removable singular point of f_4.

Indeed, we have

$$f_4(z) = \frac{z-(e^z-1)}{z(e^z-1)} = \frac{z-\left(\frac{z}{1!}+\frac{z^2}{2!}+o(z^2)\right)}{z\left(\frac{z}{1!}+\frac{z^2}{2!}+o(z^2)\right)}$$

$$= \frac{-\frac{z^2}{2}+o(z^2)}{z^2+o(z^2)} = \frac{-\frac{1}{2}+\frac{o(z^2)}{z^2}}{1+\frac{o(z^2)}{z^2}}$$

$\Rightarrow \lim_{z\to 0} f_4(z) = -\frac{1}{2}$. This implies that $z=0$ is a removable singular point of f_4 $\square$

2\. a) $$I = \int_0^{2\pi} \frac{dx}{\cos x + \sin x + 2}$$

- $z = e^{ix}$, $dz = ie^{ix}dx \Rightarrow dx = \frac{dz}{iz}$
- $\cos x = \frac{e^{ix}+e^{-ix}}{2} = \frac{z^2+1}{2z}$
- $\sin x = \frac{e^{ix}-e^{-ix}}{2i} = \frac{z^2-1}{2iz}$

Thus,
$$I = \int_{|z|=1} \frac{\frac{dz}{iz}}{\frac{z^2+1}{2z} + \frac{z^2-1}{2iz} + 2}$$
$$= 2 \cdot \int_{|z|=1} \frac{dz}{(1+i)z^2 + 4iz + i - 1)}$$

$(1+i)z^2 + 4iz + i - 1 = 0$

$$\Delta' = (2i)^2 - (1+i)(i-1) = -4 + 2 = -2$$

$$\Leftrightarrow \begin{cases} z_1 = \frac{-2i + i\sqrt{2}}{1+i} \in \Delta = \{|z|<1\} \\ z_2 = \frac{-2i - i\sqrt{2}}{1+i} \notin \bar{\Delta} = \{|z|\leq 1\} \end{cases}$$

$$\operatorname{Res}_{z=z_1} \frac{1}{(1+i)z^2+4iz+i-1} = \frac{1}{2(1+i)z_1 + 4i} = 1/2i\sqrt{2} = \frac{1}{2i\sqrt{2}}$$

By Cauchy's residue theorem, we get

$$I = 2 \cdot 2\pi i \operatorname{Res}_{z=z_1}(f) = 4\pi i (2i\sqrt{2})^{-1} = \frac{4\pi i}{2i\sqrt{2}} = \pi\sqrt{2}$$

$$I = \pi\sqrt{2}$$

b) $I = \int_0^{\pi} \frac{\cos(2x)}{1+2a\cos x + a^2} dx$, $-1 < a < 1$

We have

① $\int_{\pi}^{2\pi} \frac{\cos(2x)}{1+2a\cos x + a^2} dx \overset{t = x-\pi}{=\!=} \int_0^{\pi} \frac{\cos(2t)}{1-2a\cos(t) + a^2} dt$

② $\int_0^{\pi} \frac{\cos(2t)}{1-2a\cos(t)+a^2} dt \overset{x=\pi-t}{=\!=} \int_0^{\pi} \frac{\cos(2x)}{1+2a\cos x + a^2} dx.$

$\Rightarrow \int_0^{2\pi} \frac{\cos(2x)}{1+2a\cos x+a^2} dx = \int_0^{\pi} \frac{\cos(2x)}{1+2a\cos x+a^2} dx + \int_0^{2\pi} \frac{\cos(2x)}{1+2a\cos x+a^2} dx$

$= 2\int_0^{\pi} \frac{\cos(2x)}{1+2a\cos x+a^2} dx = 2I$ (by ① and ②)

Let $z = e^{ix}$, $dz = ie^{ix}dx \Rightarrow dx = \frac{dz}{iz}$

$\cos x = \frac{e^{ix}+e^{-ix}}{2} = \frac{z^2+1}{2z}$, $\cos(2x) = \frac{e^{2ix}+e^{-2ix}}{2}$

$= \frac{z^4+1}{2z^2}$

$\Rightarrow I = \frac{1}{2}\int_{|z|=1} \frac{\frac{z^4+1}{2z^2}}{1+2a\cdot\frac{z^2+1}{2z}+a^2} \frac{dz}{iz}$

$= \frac{1}{4i}\int_{|z|=1} \frac{z^4+1}{z^2(az^2+(1+a^2)z+a)} dz$

$* \boxed{a \neq 0}$

Note: $az^2+(1+a^2)z+a=0 \Leftrightarrow \left[\begin{array}{l} z_1 = -a, \quad |z_1|<1 \\ z_2 = -\frac{1}{a}, \quad |z_2|>1 \end{array}\right.$

$(a \neq 0)$

$f(z) := \frac{z^4+1}{z^2(az^2+(1+a^2)z+a^2)}$

$$\operatorname*{Res}_{z=0}(f) = \left(\frac{z^4+1}{az^2+(1+a^2)z+a}\right)'_{z=0}$$

$$= \left.\frac{4z^3(az^2+(1+a^2)z+a) - (z^4+1)(2az+1+a^2)}{(az^2+(1+a^2)z+a)^2}\right|_{z=0}$$

$$= -\frac{1+a^2}{a^2} = -\frac{1+a^2}{a^2}$$

$$\operatorname*{Res}_{z=-a}(f) = \left.\frac{(z^4+1)/z^2}{(az^2+(1+a^2)z+a)'}\right|_{z=-a}$$

$$= \frac{a^4+1}{a^2(2a(-a)+1+a^2)} = \frac{a^4+1}{a^2(1-a^2)}$$

Therefore,

$$I = \frac{1}{4i}\, 2\pi i \left[\operatorname*{Res}_{z=0}(f) + \operatorname*{Res}_{z=-a}(f)\right]$$

$$= \frac{\pi}{2}\left[-\frac{1+a^2}{a^2} + \frac{a^4+1}{a^2(1-a^2)}\right]$$

$$= \frac{\pi}{2}\,\frac{-(1+a^2)(1-a^2)+a^4+1}{a^2(1-a^2)}$$

$$= \frac{\pi}{2}\cdot\frac{2a^4}{a^{\cancel{4}2}(1-a^2)}$$

$$= \pi\cdot\frac{a^2}{1-a^2} \quad (a \neq 0)$$

* $\boxed{a=0}$

Note : if $a=0$, then $I = \int_0^{\pi} \cos(2x)\,dx = 0$.

Hence, $I = \pi\cdot\dfrac{a^2}{1-a^2}, \quad -1 < a < 1$.

c) $I = \int_0^{\pi} \cos^{2n} x \, dx = \int_0^{\pi} \left(\frac{1 + \cos 2x}{2}\right)^n dx$

$$= \frac{1}{2}\int_0^{2\pi} \left(\frac{1 + \cos t}{2}\right)^n dt$$

Let $z = e^{it}$, $dz = ie^{it} dt \Rightarrow dt = \frac{dz}{iz}$

$$\cos t = \frac{e^{it} + e^{-it}}{2} = \frac{z^2+1}{2z}$$

$$\Rightarrow I = \frac{1}{2}\int_{|z|=1} \left(\frac{1 + \frac{z^2+1}{2z}}{2}\right)^n \frac{dz}{iz}$$

$$= \frac{1}{2^{2n+1} \cdot i} \cdot \int_{|z|=1} \frac{(z+1)^{2n}}{z^{n+1}} dz$$

$$= \frac{1}{2^{2n+1} \cdot i} \cdot 2\pi i \operatorname*{Res}_{z=0} \left(\frac{(z+1)^{2n}}{z^{n+1}}\right)$$

Since $(z+1)^{2n} = 1 + \binom{2n}{1} z + \cdots + \binom{2n}{n} z^n + \cdots + \binom{2n}{2n} z^{2n}$,

$$\operatorname*{Res}_{z=0} \frac{(z+1)^{2n}}{z^{n+1}} = \binom{2n}{n} = \frac{(2n)!}{n!\, n!}.$$

Therefore,

$$I = \frac{\pi}{2^{2n}} \cdot \frac{(2n)!}{(n!)^2}.$$

d) $I = \int_0^{2\pi} \sin^{2n}(x)\,dx$

- $z = e^{ix}$, $dz = ie^{ix}dx \Rightarrow dx = \dfrac{dz}{iz}$
- $\sin x = \dfrac{e^{ix}-e^{-ix}}{2i} = \dfrac{z^2-1}{2iz}$

$$\Rightarrow I = \int_{|z|=1} \left(\frac{z^2-1}{2iz}\right)^{2n} \cdot \frac{dz}{iz}$$

$$= \int_{|z|=1} \frac{1}{2^{2n}(-1)^n i} \frac{(z^2-1)^{2n}}{z^{2n+1}}\,dz$$

$$= 2\pi i \cdot \frac{1}{2^{2n}(-1)^n i} \operatorname*{Res}_{z=0} \frac{(z^2-1)^{2n}}{z^{2n+1}}$$

$$= \frac{\pi}{2^{2n-1}(-1)^n} \cdot \operatorname*{Res}_{z=0} \frac{(z^2-1)^{2n}}{z^{2n+1}}$$

Since, $(z^2-1)^{2n} = (1-z^2)^{2n} = 1 - \binom{2n}{1} z^2 + \cdots + (-1)^n \binom{2n}{n} z^{2n} + \cdots + z^{4n}$,

we have $\operatorname*{Res}_{z=0} \dfrac{(z^2-1)^{2n}}{z^{2n+1}} = (-1)^n \binom{2n}{n} = (-1)^n \cdot \dfrac{(2n)!}{n!\,n!}$.

So, $I = \dfrac{\pi}{2^{2n-1}(-1)^n} (-1)^n \cdot \dfrac{(2n)!}{(n!)^2} = \pi \cdot \dfrac{(2n)!}{(n!)^2\, 2^{2n-1}}$.

3. a) $z^{2014} - 6z^{2013} + 3z^{2012} - 1$, $D = \{ z \in \mathbb{C} \mid |z| < 1 \}$

- $f(z) := -6z^{2013}$ has 2013 zeros in D
- $g(z) := z^{2014} + 3z^{2012} - 1$.

We have $\left|\dfrac{g(z)}{f(z)}\right|_{z \in \partial D} = \dfrac{|z^{2014} + 3z^{2012} - 1|}{6\,|z|^{2013}}\Bigg|_{|z|=1} \le \dfrac{|z|^{2014} + 3|z|^{2012} + 1}{6\,|z|^{2013}}\Bigg|_{|z|=1}$

$$\le \frac{1+3+1}{6} = \frac{5}{6} < 1 .$$

By Rouché's theorem,

$z^{2014} - 6z^{2013} + 3z^{2012} - 1 = f(z) + g(z)$ has 2013 zeros in D.

b) $z^3 + 4z^2 - 2z + 2$, $D = \{ |z| < 3 \}$

- $f(z) := 4z^2$ has 2 zeros in D
- $g(z) := z^3 - 2z + 2$.

We have $\left|\dfrac{g(z)}{f(z)}\right|_{|z|=3} \le \dfrac{|z|^3 + 2\cdot|z| + 2}{4|z|^2}\Bigg|_{|z|=3} = \dfrac{27 + 2\cdot 3 + 2}{36}$

$$< \frac{35}{36} < 1 .$$

Hence, by Rouché's theorem we get

$z^3 + 4z^2 - 2z + 2 = f(z) + g(z)$ has 2 zeros in D.

4) a) $P(z) = 2z^5 - 5z^2 + z + 1 = 0$

$D_1 = \{ z \in \mathbb{C} \mid |z| < 1 \}$

$D_2 = \{ z \in \mathbb{C} \mid |z| < 2 \}$

(i) in D_1.

$f(z) := -5z^2$ has 2 zeros in D_1

$g(z) := 2z^5 + z + 1$

$$\left|\frac{g(z)}{f(z)}\right|_{z \in \partial D_1} = \left.\frac{|2z^5 + z + 1|}{|-5z^2|}\right|_{|z|=1} \leq \left.\frac{2 \cdot |z|^5 + |z| + 1}{5 \cdot |z|^2}\right|_{|z|=1} = \frac{4}{5} < 1$$

By Rouché's theorem, $P(z) = f(z) + g(z)$ has 2 zeros in D_1

(ii) in D_2

$f(z) := 2z^5$ has 5 zeros in D_2

$g(z) := -5z^2 + z + 1$

$$\left|\frac{g(z)}{f(z)}\right|_{z \in \partial D_2} \leq \left.\frac{5|z|^2 + |z| + 1}{2|z|^5}\right|_{|z|=2} = \frac{5 \cdot 4 + 2 + 1}{2 \cdot 2^5} = \frac{23}{64} < 1$$

By Rouché's theorem, we obtain that

$P(z) = f(z) + g(z)$ has 5 zeros in D_2.

We note that $|P(z)| \geq 5|z|^2 - 2|z|^5 - |z| - 1$

$\geq 5 \cdot 1 - 2 \cdot 1 - 1 - 1 = 1 > 0$

for $\forall z \in \partial D_1$. This implies that P has no zeros in ∂D_1.

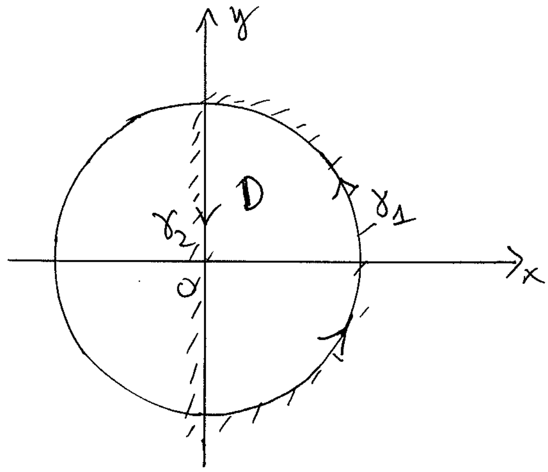
Therefore, $P(z)$ has $5 - 2 = 3$ zeros in $D_2 \setminus \overline{D_1} = \{1 < |z| < 2\}$

4 b) $0.9e^{-z}+1=2z \quad (1)$, $D=\{|z|<1, \operatorname{Re} z>0\}$

Let

$f(z):=1-2z$

$g(z):=0.9e^{-z}$



$\gamma_1=\{|z|=1, \operatorname{Re} z\geq 0\}$

$\gamma_2=\{x=0, -1\leq y\leq 1\}$.

Then we have

+ (1) $\Leftrightarrow f(z)+g(z)=0$

+ $\partial D=\gamma_1\cup\gamma_2$.

Moreover, $\left|\frac{g(z)}{f(z)}\right|_{z\in\gamma_1}\leq\left.\frac{0.9\,|e^{-z}|}{2|z|-1}\right|_{z\in\gamma_1}=\left.\frac{0.9\,e^{-\operatorname{Re} z}}{2|z|-1}\right|_{z\in\gamma_1}$

$\leq\frac{0.9\cdot 1}{2-1}=\frac{0.9}{1}<1$

and $\left|\frac{g(z)}{f(z)}\right|_{z\in\gamma_2}\leq\left.\frac{0.9\,|e^{-z}|}{|1-2z|}\right|_{z\in\gamma_2}=\left.\frac{0.9\,e^{0}}{|1-2iy|}\right|_{z=iy\in[-1,1]}$

$\leq\frac{0.9}{\sqrt{1+(2y)^2}}<\frac{0.9}{1}<1$.

Therefore, $\left|\frac{g(z)}{f(z)}\right|_{z\in\partial D}<1$.

Since $f(z)=1-2z$ has 1 zero $(z=\frac{1}{2})$ in D, the equation (1) has 1 solution in D.

Homework III (due to Tuesday 10 Dec., 2013)

Math 210: Applied Complex Varibles, Fall 2013

1. (30 points) Find the linear fractional transformation that maps the points $z_1 = 1, z_2 = i, z_3 = -1$ onto the points $w_1 = 2, w_2 = i, w_3 = -2$.

2. (30 points) Find the image of the upper half plane $H = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$ under the transformation $w = \dfrac{i-z}{i+z}$.

3. (30 points) Find the image of the semi-infinite trip $D = \{z \in \mathbb{C} \mid 0 < \text{Re}(z) < \pi/2,\ \text{Im}(z) > 0\}$ under the map $w = \sin(z)$.

4. (10 points) Find a one to one mapping $w = f(z)$ that maps the domain $D = \{z \in \mathbb{C} \mid \text{Im}(z) > 0,\ |z| > 1\} \setminus [i, 2i]$ onto the upper half plane $H = \{w \in \mathbb{C} \mid \text{Im}(w) > 0\}$.

————End————

Solutions / answers for Homework III

1. the linear fractional transform that maps $z_1 = 1, z_2 = i, z_3 = -1$ onto $w_1 = 2, w_2 = i, w_3 = -2$ is given by

$$\frac{(w-2)(i+2)}{(w+2)(i-2)} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$\Leftrightarrow \quad \frac{w-2}{w+2} \cdot \left(-\frac{4i+3}{5}\right) = \frac{z-1}{z+1}(-i)$$

$$\Leftrightarrow \quad w = \frac{6z-2i}{-iz+3}.$$

2. $H = \{ z \in \mathbb{C} \mid \operatorname{Im}(z) > 0 \}$

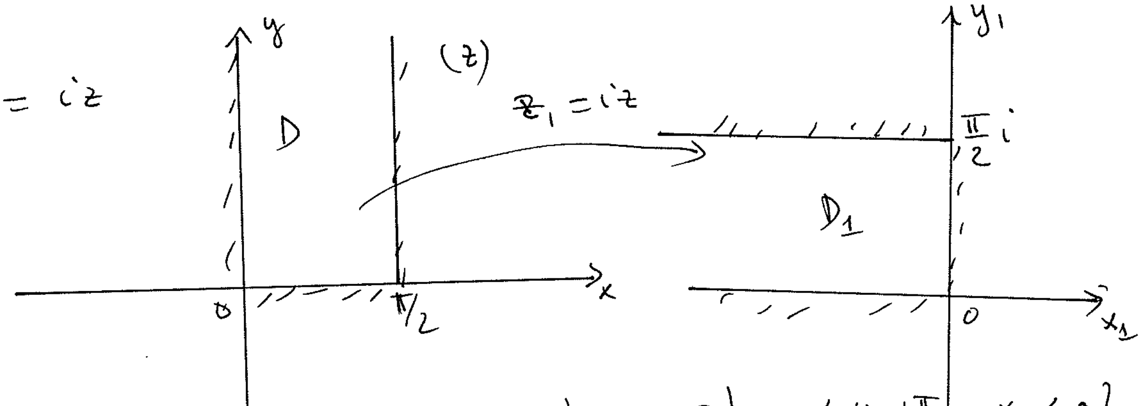
$$w = \frac{i-z}{i+z} = -\frac{z-i}{z-(-i)} = e^{\pi i}\frac{z-i}{z-\bar{i}}$$

$\Rightarrow$ this maps H onto the unit disk $\mathbb{D} = \{ w \in \mathbb{C} \mid |w| < 1 \}$

3. $D = \{ z \in \mathbb{C} \mid 0 < \operatorname{Re} z < \frac{\pi}{2}, \operatorname{Im} z > 0 \}$

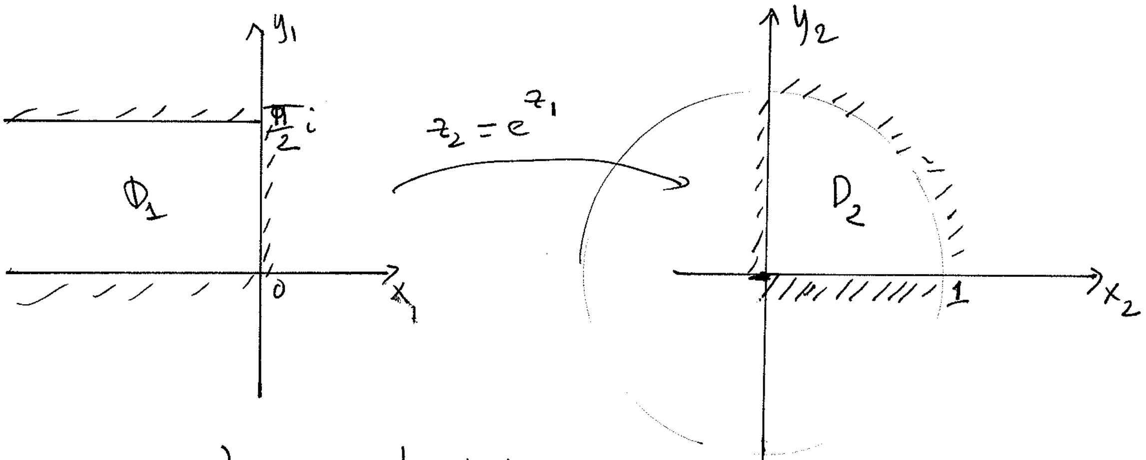
$$w = \sin z = \frac{e^{iz} - e^{-iz}}{2i} = \frac{1}{2}\left(\frac{e^{iz}}{i} + \frac{1}{\frac{e^{iz}}{i}} \right)$$

i) $z_1 = iz$



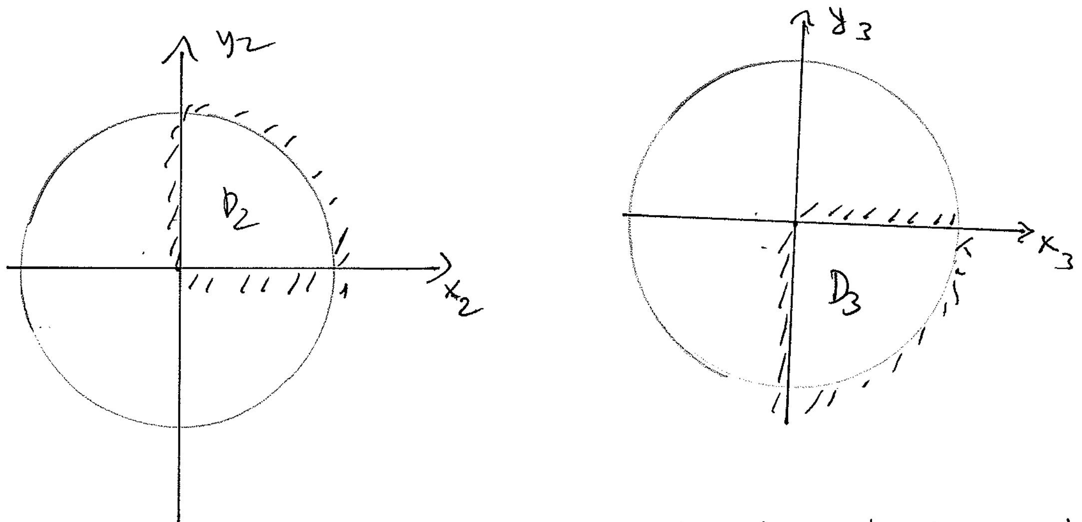
$D_1 = \{ z_1 \in \mathbb{C} \mid 0 < y_1 < \frac{\pi}{2}, x_1 < 0 \}$

ii) $z_2 = e^{z_1}$



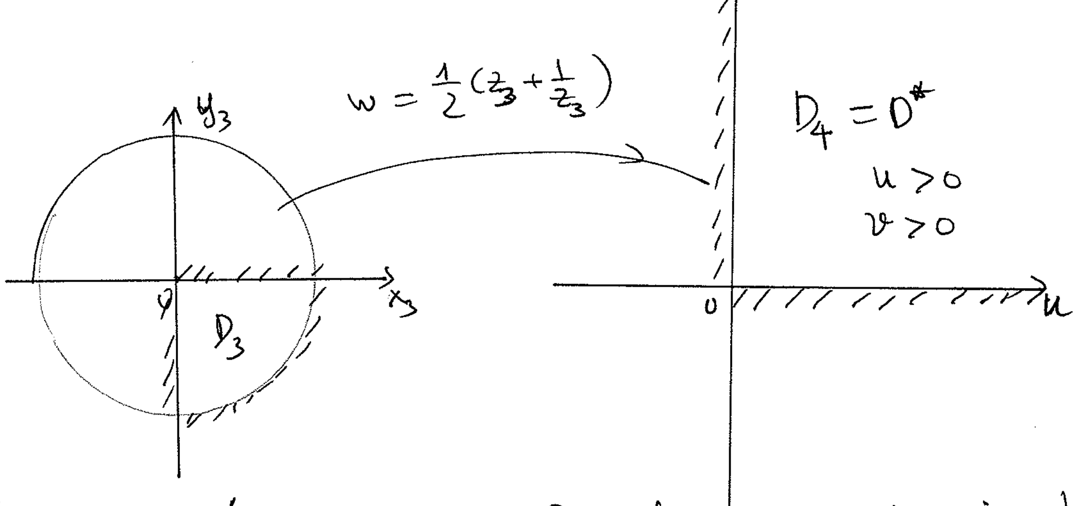
$$D_2 = \{ z_2 \in \mathbb{C} \mid |z_2| < 1, x_2 > 0, y_2 > 0 \}$$

iii) $z_3 = \dfrac{z_2}{i}$ maps D_2 onto $D_3 = \{ z_3 \in \mathbb{C} \mid |z_3| < 1, x_3 > 0, y_3 < 0 \}$



(iv) $w = \frac{1}{2}\left(z_3 + \frac{1}{z_3}\right)$ maps D_3 onto $\cancel{D_4} \neq D_4^* = \{ w \in \mathbb{C} \mid u > 0, v > 0 \}$

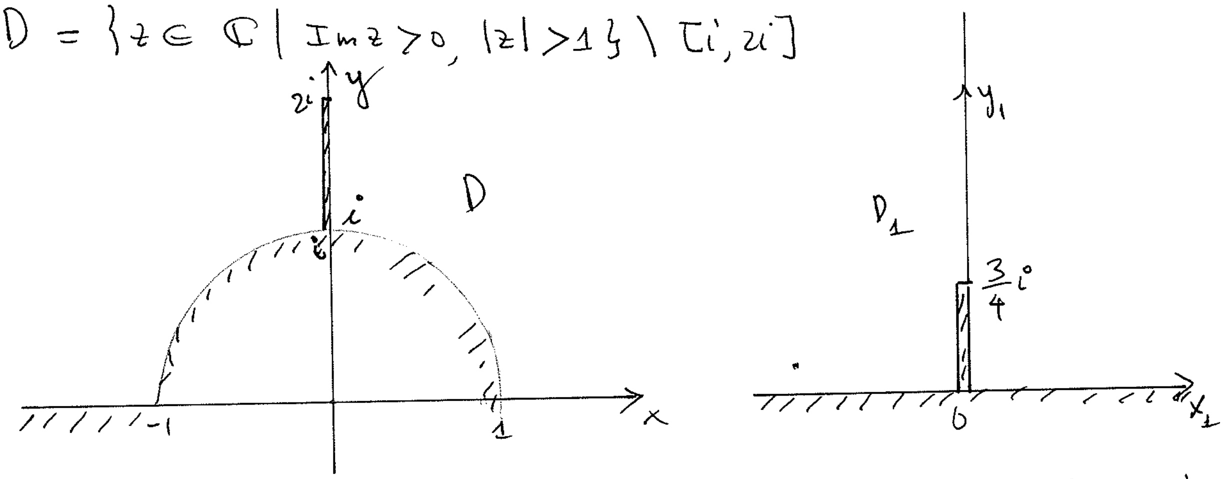
$w = \sin z$



Therefore, D^* = the image of D under $w = \sin z$ is given by

$$D^* = \{ w \in \mathbb{C} \mid u > 0, v > 0$$

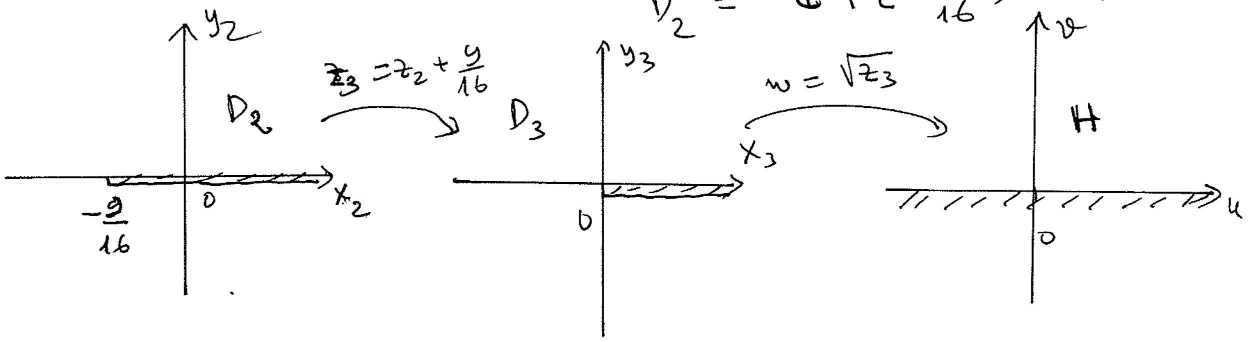
4. $D = \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0, \ |z| > 1 \} \setminus [i, 2i]$



i) $z_1 = \frac{1}{2}\left(z + \frac{1}{z}\right)$ maps D onto $D_1 = \{ z_1 \in \mathbb{C} \mid x_1 > 0 \} \setminus [0, \frac{3i}{4}]$

ii) $z_2 = z_1^2$ maps D_1 onto $D_2 = \{ z_2 \in \mathbb{C} \} \setminus [-\frac{9}{16}, +\infty)$

$$D_2 = \mathbb{C} \setminus [-\tfrac{9}{16}, +\infty)$$



iii) $z_3 = z_2 + \frac{9}{16}$ maps D_2 onto $D_3 = \mathbb{C} \setminus [0, +\infty)$

iv) $w = \sqrt{z_3}$ maps D_3 onto $H = \{ w \in \mathbb{C} \mid \operatorname{Im}(w) > 0 \}$

Hence, the map $w = \sqrt{\left[\frac{1}{2}\left(z + \frac{1}{z}\right)\right]^2 + \frac{9}{16}}$ maps D one-to-one

$$w = \sqrt{\frac{1}{4}\left(z^2 + \frac{1}{z^2}\right) + \frac{17}{16}}$$

to $\mathbb{H}$.