On the tangential holomorphic vector fields vanishing at an infinite type point

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References

This talk is based on the followings:


Purpose of the talk

We consider a real hypersurface in \( \mathbb{C}^2 \) around the origin which is of infinite type in the sense of D’Angelo.

The purpose of this talk is twofold.

1. *Characterization of real hypersurfaces of infinite type.*

2. *Characterization of holomorphic vector fields tangent to a real hypersurface.*
Vanishing order

Definition

Let $M$ be a germ of a smooth real hypersurface in $\mathbb{C}^n$ at $p \in \mathbb{C}^n$ and $\rho$ a smooth local defining function for $M$ in some open neighborhood $U \subset \mathbb{C}^n$ of $p$, i.e., $M = \{ z \in U : \rho(z) = 0 \}$ with $d\rho(p) \neq 0$.

For any germ of a nonconstant holomorphic curve $\gamma : (\mathbb{C},0) \rightarrow (\mathbb{C}^n,p)$,

- $\nu(\gamma)$: the lowest vanishing order at 0 of all components of $\gamma(t) - \gamma(0)$.
- $\nu(f)$: the vanishing order of a function $f$ at 0, $f : \mathbb{C} \supset U(0) \rightarrow \mathbb{R}$, $U(0)$ is a n.b.h. of 0 in $\mathbb{C}$.
- $\nu(f) = +\infty \iff \frac{\partial^{j+k}}{\partial z^j \partial \bar{z}^k} f(0) = 0$, $\forall j, k = 0,1,\ldots$. 
**D’Angelo type**

**Definition**

The *D’Angelo type of $M$ at $p$* is defined by

$$\tau(M, p) := \sup_{\gamma} \frac{\nu(\rho \circ \gamma)}{\nu(\gamma)},$$

where the supremum is taken over all germs of nonconstant holomorphic curves $\gamma : (\mathbb{C}, 0) \to (\mathbb{C}^n, p)$. 
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where the supremum is taken over all germs of nonconstant holomorphic curves $\gamma : (\mathbb{C}, 0) \to (\mathbb{C}^n, p)$.

We say that $p$ is a point of finite type if $\tau(M, p) < \infty$ and of infinite type if otherwise.
Examples

Let $E_{1,m} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^{2m} = 1\}$, where $m = 1, 2, \ldots$. Then $\tau(E_{1,m}, (1, 0)) = 2m$. 
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Then $\tau(E_{1,m}, (1, 0)) = 2m$.

Let $E_{1,\infty} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + P(z_2) = 1\}$, where $P(z_2) = 2e^{-1/|z_2|^2}$ if $z_2 \neq 0$ and $P(0) = 0$.

Then $\tau(E_{1,\infty}, (1, 0)) = +\infty$. 
Characterization of real hypersurfaces of infinite type

Let $M$ be a $C^\infty$-smooth real hypersurface in $\mathbb{C}^2$ given locally by

$$M = \{(z_1, z_2) \in \mathbb{C}^2 : \rho(z_1, z_2) = \text{Re} \ z_1 + P(z_2) + \text{Im} \ z_1 Q(z_2, \text{Im}z_1) = 0\}$$

for some $C^\infty$-smooth functions $P$ and $Q$ with $P(0) = dP(0) = 0$ and $Q(0, 0) = 0$. 

**Theorem 1**

Suppose that $P$ contains no harmonic terms, i.e.,

$$\partial^N N \partial z_{2^N} P(0) = 0 \quad \forall \ N \in \mathbb{N}.$$ 

Then $\tau(M, 0) = +\infty$ if and only if $\nu(P) = +\infty$. 

Characterization of real hypersurfaces of infinite type

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**Theorem 1**

Suppose that $P$ contains no harmonic terms, i.e., $\frac{\partial^N}{\partial z_2^N} P(0) = 0 \ \forall N \in \mathbb{N}$.

Then $\tau(M, 0) = +\infty$ if and only if $\nu(P) = +\infty$. 
Characterization of real hypersurfaces

In [3, Math. Res. Lett. (2005)], Martin Kolar showed that if $\tau(M, p) < +\infty$, then after the change of variables the defining function has form

$$
\rho(z_1, z_2) = \text{Re} z_1 + H(z_2) + o(|\text{Im} z_1|, |z_2|^k),
$$

where $H(z_2)$ is a real homogeneous polynomial of degree $k \ (= \tau(M, p))$ and $H$ contains no harmonic terms (i.e., $az^k, b\bar{z}^k$).
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For the case that $\tau(M,p) = \infty$ we also obtain the following proposition.

**Proposition 1**

If $\tau(M,0) = +\infty$, then there exists a sequence $\{a_n\}_{n=2}^{\infty} \subset \mathbb{C}$ s.t. for each $N \geq 2$ we have

$$\nu(\rho \circ \varphi_N(\zeta)) \geq N,$$

where $\varphi_N$ is a regular holomorphic curve given on a neighborhood of $t = 0$ in $\mathbb{C}$ by

$$z_1(t) = - \sum_{j=2}^{N} a_j t^j; \quad z_2(t) = t.$$
Characterization of real hypersurfaces

Remark 1

If the series \( \sum_{j=2}^{\infty} a_j z^j \) is convergent on a neighborhood of \( z = 0 \) in \( \mathbb{C} \), then \( \nu(\rho \circ \varphi_{\infty}) = +\infty \), where \( \varphi_{\infty} \) is the holomorphic curve given on a neighborhood of \( t = 0 \) in \( \mathbb{C} \) by

\[
\begin{align*}
z_1(t) &= -\sum_{j=2}^{\infty} a_j t^j; \\
z_2(t) &= t.
\end{align*}
\]
Characterization of real hypersurfaces

Remark 1
If the series $\sum_{j=2}^{\infty} a_j z^j$ is convergent on a neighborhood of $z = 0$ in $\mathbb{C}$, then $\nu(\rho \circ \varphi_\infty) = +\infty$, where $\varphi_\infty$ is the holomorphic curve given on a neighborhood of $t = 0$ in $\mathbb{C}$ by

$$z_1(t) = -\sum_{j=2}^{\infty} a_j t^j; \quad z_2(t) = t.$$ 

Question
Does there exist a nontrivial holomorphic curve $\varphi_\infty : \Delta \to \mathbb{C}^2$ such that $\nu(\rho \circ \varphi_\infty) = +\infty$?
Characterization of real hypersurfaces

Remark 2

- $M$ is real-analytic and $\tau(M, p) = +\infty \Rightarrow \exists \varphi_\infty$ such that $\nu(\rho \circ \varphi_\infty) = +\infty$. (cf. J. P. D’Angelo [2], 1992.)
Characterization of real hypersurfaces

Remark 2

- $M$ is real-analytic and $\tau(M, p) = +\infty \Rightarrow \exists \varphi_\infty$ such that $\nu(p \circ \varphi_\infty) = +\infty$. (cf. J. P. D’Angelo [2], 1992.)

- In general, the answer is ”no”. The following example gives a negative answer.
Example

For $n = 2, 3, \ldots$, denote by

$$g_n(z) = \frac{1}{z^n - a_n} + \frac{1}{a_n}$$

for $|z| < 1/n$, where $a_n = 2/n^n$. Then $g_n$ is holomorphic on $\{|z| < 1/n\}$ with $\nu(g_n) = n$.

For each $n = 2, 3, \ldots$ denote by $\tilde{f}_n(z)$ a $C^\infty$-smooth function on $\mathbb{C}$ such that

$$\tilde{f}_n(z) = \begin{cases} \text{Re} \left( g_n(z) \right) & \text{if } |z| < 1/(n + 1) \\ 0 & \text{if } |z| > 1/n. \end{cases}$$
Example

Let

$$\lambda_n := \max \{1, \| \frac{\partial^{k+j}}{\partial z^k \partial \bar{z}^j} \tilde{f}_n \|_{\infty} ; j, k \in \mathbb{N}, k + j \leq n\},$$

$$f_n(z) := \frac{1}{n^n \lambda_n^n} \tilde{f}_n(\lambda_n z),$$

and

$$f(z) := \sum_{n=2}^{\infty} f_n(z).$$

We now consider the hypersurface $M$ defined locally by

$$M = \{(z_1, z_2) \in \mathbb{C}^2 : \rho = \text{Re} \ z_1 + f(z_2) = 0\}.$$ 

Then $\tau(M, 0) = +\infty$ and there is no regular holomorphic curve $\varphi_\infty(t) = (h(t), t)$, where $h$ is a holomorphic function defined on a neighborhood of 0 in $\mathbb{C}$, s.t. $\nu(\rho \circ \varphi_\infty) = +\infty$. 
Holomorphic vector fields tangent to a real hypersurface

Let \( H = (h_1, h_2) = h_1(z_1, z_2) \frac{\partial}{\partial z_1} + h_2(z_1, z_2) \frac{\partial}{\partial z_2} \) be a holomorphic vector field on a neighborhood of \((0, 0)\) in \( \mathbb{C}^2 \).

We say that \( H \) is tangent to \( M \) if

\[
\text{Re } H(\rho(z)) = 0, \quad \forall \ z \in M.
\]
Holomorphic vector fields tangent to a real hypersurface

Let $H = (h_1, h_2) = h_1(z_1, z_2) \frac{\partial}{\partial z_1} + h_2(z_1, z_2) \frac{\partial}{\partial z_2}$ be a holomorphic vector field on a neighborhood of $(0,0)$ in $\mathbb{C}^2$. We say that $H$ is tangent to $M$ if

$$\text{Re } H(\rho(z)) = 0, \forall z \in M.$$

In [1, J. Math. Anal. Appl. (2012)], Jisoo Byun, Jae-Cheon Joo and Minju Song showed that if $\rho(z_1, z_2) = \rho(z_1, |z_2|)$ then $H$ is tangent to $M$ if and only if $H(z) = i\alpha z_2 \frac{\partial}{\partial z_2}, \alpha \in \mathbb{R}$. 
Holomorphic vector fields tangent to a real hypersurface

Theorem 2

If a hypersurface germ \((M, 0)\) is defined by the equation

\[
\rho(z) := \rho(z_1, z_2) = \text{Re } z_1 + P(z_2) + (\text{Im } z_1)Q(z_2, \text{Im } z_1) = 0,
\]

satisfying the conditions:

(i) \(P(z_2) > 0\) for any \(z_2 \neq 0\);

(ii) \(P\) vanishes to infinite order at \(z_2 = 0\);

(iii) \(\frac{\partial^N Q(z_2, 0)}{\partial z_2^N} \bigg|_{z_2=0} = 0\) for every positive integer \(N\);

then any holomorphic vector field vanishing at the origin tangent to \((M, 0)\) is either identically zero, or of the form \(i\beta z_2 \frac{\partial}{\partial z_2}\) for some non-zero real number \(\beta\), in which case it holds that \(\rho(z_1, z_2) = \rho(z_1, |z_2|)\).
Remark 3

There exists a hypersurface germ \((M, 0)\) satisfying the hypotheses of Theorem 2 except the condition \((iii)\), which admits a nontrivial holomorphic tangent vector field with both \(\partial/\partial z_1\) and \(\partial/\partial z_2\) present in the expression nontrivially.
Example

Let $M$ be the real hypersurface in $\Delta^2 \subset \mathbb{C}^2$ defined by

$$M = \{(z_1, z_2) \in \Delta^2 : \rho(z_1, z_2) = \text{Re } z_1 + P(z_2) + (\text{Im } z_1)Q(z_2) = 0\},$$

where $\Delta := \{z \in \mathbb{C} : |z| < 1\}$, $P$ and $Q$ are given as follows:

$$Q(z_2) = \tan((\text{Im } z_2)^2)$$

and

$$P(z_2) = \begin{cases} 
\exp \left( -\frac{1}{|z_2|^2} + \frac{1}{2} \text{Im } (z_2^2) - \log |\cos((\text{Im } z_2)^2)| \right) & \text{if } 0 < |z_2| < 1 \\
0 & \text{if } z_2 = 0.
\end{cases}$$

Then the holomorphic vector field $H$ defined by

$$H = z_1z_2^2 \frac{\partial}{\partial z_1} + iz_2 \frac{\partial}{\partial z_2}$$

is tangent to the hypersurface $M$. 
Thank you for your attention!